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Metabelian techniques in knot concordance

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Metabelian techniques in knot concordance

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Metabelian techniques in knot concordance

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This dissertation lies in the field of knot concordance, the study of 4-dimensional properties of knots. We give four distinct results, which are united by their mutual reliance on concordance invariants associated to metabelian covers of certain 3-manifolds. First, we give some examples of 2-bridge knots for which twisted Alexander polynomials but not Casson-Gordon signatures obstruct sliceness. We then use Casson-Gordon signatures to give a complete characterization of the topologically slice odd 3-strand pretzel knots, and an almost complete characterization of the topologically slice even 3-strand pretzel knots. Next, we describe large infinite families of 4-strand pretzel knots which are not even topologically slice, despite being positive mutants of ribbon knots. We conclude by proving that given any patterns P and Q of opposite winding number, for any $n \geq 0$ there exists a knot K such that the minimal genus of a cobordism between $P(K)$ and $Q(K)$ is at least n. This completes the argument, partially established in [7], that two patterns are a finite distance apart in their action on concordance if and only if they have the same algebraic winding number.

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Chapter 1

Introduction

1.1 Background

The fundamental question of knot concordance is when is a knot *slice*: that is, considering the 3-dimensional sphere as the boundary of the 4-dimensional ball, when does a knot bound a 2-dimensional disc in that ball? This question turns out to capture a great deal of the complexity of 4-dimensional topology.

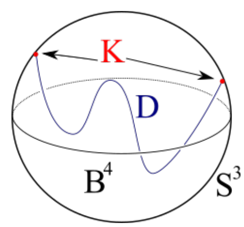

Figure 1.1: A knot K in the 3-sphere S^3 bounding a disc D in the 4-ball B^4.

For any knot K in S^3 we can observe that $\mathrm{Cone}(S^3, K) = (B^4, D)$, where D is a topologically embedded disc with $\partial D = K$. In order to avoid a trivial definition, we therefore impose one of the following additional requirements on our embedddings.

Definition 1.1.1. A knot K in S^3 is *smoothly (topologically) slice* if there is a smoothly (locally flatly[1]) embedded disc D in B^4 with $\partial D = K$.

The groundbreaking work of Freedman and Donaldson implies that these two notions of sliceness do not coincide: Freedman's Disc Embedding Theorem can be used to show that any knot with trivial Alexander polynomial is topologically slice, whereas Donaldson's gauge theoretic work implies that many of these knots are not smoothly slice. In fact, the existence of topologically slice, smoothly non-slice knots can be used to construct *exotic* $\mathbb{R}^4$s: manifolds homeomorphic and yet not diffeomorphic to $\mathbb{R}^4$.

The idea of sliceness can be extended to give an equivalence relation on knots in S^3: two knots K_0 and K_1 are *concordant* if they cobound an embedded annulus in $S^3 \times I$. Note that a knot is slice if and only if it is concordant to the unknot, and that concordance is an equivalence relation. We let $\mathcal{C}$ denote the collection of knots in S^3 modulo concordance (when we want to emphasize category, we write $\mathcal{C}^s$ and $\mathcal{C}^t$ for the collections of the smooth and topological concordance classes of knots, respectively.) Given a knot K, we let the knot $-K := m(K^r)$ denote the knot obtained by reversing the orientations of both K and S^3. Then $K \# -K$ is always slice and so the monoid structure on the collection of knots modulo isotopy becomes a group structure $\mathcal{C}$. The group theoretic structure of $\mathcal{C}$ is not very well understood: while we know that $\mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty \leq \mathcal{C}$, it remains possible that, for example, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are also subgroups of $\mathbb{C}$.

[1] An embedding $i\colon \mathbb{D}^2 \to B^4$ is *locally flat* if for every point $x \in \mathbb{D}^2$ there is a neighborhood U of x and a neighborhood V of $i(x)$ such that the pair $(V, i(U))$ is homeomorphic to the standard pair $(\mathrm{int}(\mathbb{D}^4), \mathrm{int}(\mathbb{D}^2))$. It is a consequence of work of Freedman that such an embedding is *globally flat*, i.e. that there is an neighborhood W of $i(\mathbb{D}^2)$ in B^4 such that $(W, i(\mathbb{D}^2))$ is homeomorphic to the standard pair $(D^2 \times \mathrm{int}(\mathbb{D}^2), \mathbb{D}^2 \times \{0\})$.

One can define the sliceness and concordance of n-knots, embedded S^ns in S^{n+2}, analogously for $n > 1$, and we similarly obtain a group $\mathcal{C}_n$. The following result of Kervaire shows that one should restrict to odd n.

Theorem 1.1.2 (Kervaire [22]). *When n is even, every n-knot is slice.*

In contrast to the classical concordance group $\mathcal{C} = \mathcal{C}_1$, a great deal about $\mathcal{C}_n$ is known when $n > 1$. The oldest concordance invariants are the so-called *algebraic* concordance invariants, the definitions of which immediately extend to all $n \in \mathbb{N}$. One can use these invariants to define a homomorphism from $\mathcal{C}_n$ to $\mathcal{G}_n$, the *n-dimensional algebraic concordance group*. The following result of Levine summarizes the basic properties of $\mathcal{G}_n$.

Theorem 1.1.3 (Levine [31, 30]). *For odd n, $\mathcal{G}_n \cong \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}^\infty$. Also,*

1. *The map $\mathcal{C}_n \to \mathcal{G}_n$ is an isomorphism if $n > 3$.*
2. *The map $\mathcal{C}_3 \to \mathcal{G}_3$ is an isomorphism onto an index 2 subgroup of $\mathcal{G}_3$.*
3. *The map $\mathcal{C}_1 \to \mathcal{G}_1$ is a surjection.*

In particular, for any odd $n > 1$ we have that $\mathcal{C}_n \cong \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}^\infty$. This makes it even more surprising that conjecturally there are no there are no elements of order four in $\mathcal{C}_1$!

We now briefly discuss the algebraic (also called classical or abelian) concordance invariants, deferring precise definitions. The 0-surgery of S^3 along a knot K, written M_K or $S^3_0(K)$, is the 3-manifold obtained by taking S^3, drilling out an open tubular neighborhood of K, and gluing in a solid torus so that the boundary of its meridional disc is identified with the null-homologous longitude of K. The 3-manifold M_K is a homology $S^1 \times S^2$ and in particular has an infinite cyclic cover $\widetilde{M_K}$, with a natural action by $\mathbb{Z} = \langle t \rangle$ via covering transformations. The abelian invariants of K are defined in terms of this infinite cyclic cover's homology, considered as a $\mathbb{Z}[t^{\pm 1}]$-module. In fact, it is a result of Trotter [44] that the image of a knot K in $\mathcal{G} = \mathcal{G}_1$ is equivalent to the datum of a pairing $\mathrm{Bl}\colon H_1(M_K, \mathbb{Q}[t, t^{-1}]) \times H_1(M_K, \mathbb{Q}[t, t^{-1}]) \to \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]$, up to an appropriate equivalence relation. The two most easily applied sliceness obstructions involve the Alexander polynomial and signature of a knot.

Theorem 1.1.4 (Fox-Milnor [14] and Murasugi [41]). *If K is slice, then*

1. *(Fox-Milnor [14]) The Alexander polynomial $\Delta_K(t)$ factors as $f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t, t^{-1}]$.*
2. *(Murasugi [41]) The classical signature $\sigma(K)$ equals 0.*

Knots in the kernel of $\mathcal{C} \to \mathcal{G} := \mathcal{G}_1$, which in particular satisfy the conclusions of Theorem 1.1.4, are called *algebraically slice*. The question of whether all algebraically slice knots are slice remained open until the 1970's, when Casson and Gordon [3, 4] used new metabelian invariants to show that there are many algebraically slice but not slice knots. Deferring formal definitions of the metabelian invariants we use until Chapter 2, we nevertheless give a brief overview of the approach.

Metabelian invariants come from metabelian covers of M_K in the same way that the abelian invariants come from the universal abelian cover of M_K. More concretely, any homomorphism ϕ from $\pi_1(\widetilde{M_K})$ to an abelian group induces a 'metabelian' cover M_K^ϕ, corresponding to $\ker(\phi) \subset \pi_1(\widetilde{M_K}) \subset \pi_1(M_K)$. However, in practice one considers maps coming from characters on $H_1(\Sigma_n(K)$, where $\Sigma_n(K)$ denotes the canonical nth cyclic branched cover of S^3 along K. Note that when n is a prime power $\Sigma_n(K)$ has torsion first homology and a nondegenerate linking form

$$\lambda_n \colon H_1(\Sigma_n(K) \times H_1(\Sigma_n(K) \to \mathbb{Q}/\mathbb{Z}.$$

The following result on the homology of a prime-power branched cover of S^3 along a slice knot can be proven using techniques from algebraic concordance.

Proposition 1.1.5. *Suppose that K is a slice knot. Let n be a prime power and $\Sigma_n(K)$ denote the nth cyclic branched cover of S^3 along K. Then there exists a subgroup $H \leq H_1(\Sigma_n(K))$ which is an* invariant metabolizer *for the linking form, i.e. satisfies*

(1) H is invariant under the action induced by the covering transformation on $H_1(\Sigma_n(K))$.

(2) The subgroup $H^\perp := \{g \in H_1(\Sigma_n(K)) \colon \lambda_n(g,h) = 0$ for all $h \in H\} = H$. (Equivalently, $|H|^2 = H_1(\Sigma_n(K))$ and $\lambda_n|_{H \times H} = 0$.)

Proof idea. Let Δ denote the hypothesized slice disc and $\Sigma_n(\Delta)$ denote the branched cover of B^4 along Δ. Then $H = \ker(H_1(\Sigma_n(K)) \to H_1(\Sigma_n(\Delta))$ satisfies the above. $\square$

Given $\chi \colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$, one can promote this to a homomorphism $\pi_1(M_K) \to \mathbb{Z} \ltimes \mathbb{Z}_d$. The resulting metabelian $\mathbb{Z} \ltimes \mathbb{Z}_d$ cover of M_K can be used to define metabelian analogs of the Alexander polynomial and signature.[2] The first is the *(reduced) twisted Alexander polynomial of* (K, χ), a

[2]In fact, both of these invariants are determined by the full Casson-Gordon Witt class invariant (see [4, 25]), which we will not discuss here.

Laurent polynomial with coefficients in $\mathbb{Q}(\xi_d)$ which we call $\widetilde{\Delta_K^{\chi}(t)}$. The second is the *Casson-Gordon signature of* (K, χ), which is a rational number we call $\sigma_1\tau(K, \chi)$. The following theorem is the key sliceness obstruction used throughout this thesis; we invite the reader to compare it to Theorem 1.1.4 in the context of Proposition 1.1.5.

Theorem 1.1.6 ([3],[4], [25])**.** *Suppose that K is a slice knot. Let n be a prime power. Then there is an invariant linking form metabolizer $H \leq H_1(\Sigma_n(K))$ such that for any prime power d and $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ with $\chi|_H = 0$,*

1. *The reduced twisted Alexander polynomial $\widetilde{\Delta_K^{\chi}(t)} \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ factors as $f(t)\overline{f(t)^{-1}}$ for some $f(t) \in \mathbb{Q}(\xi_{d^k})[t^{\pm 1}]$ for some $k \geq 1$.*
2. *The Casson-Gordon signature $\sigma_1\tau(K, \chi)$ equals 0.*

The *4-genus* of a knot K, written $g_4(K)$, is defined to be the minimal genus of an embedded surface in B^4 with boundary the given knot. The classical signature satisfies the inequality $|\sigma(K)| \leq 2g_4(K)$, and work of Gilmer [17] gives similar though more involved bounds from Casson-Gordon signatures.

1.2 Summary of results

The famed slice-ribbon conjecture (Problem 1.33 of the Kirby Problem List [24]) asserts that every smoothly slice knot in fact bounds a disc immersed in the 3-sphere with only 'ribbon' singularities, illustrated in Figure 1.2. Work of Lisca [32] uses Donaldson's Theorem [9] to classify the smoothly slice 2-bridge knots, giving the first family of knots for which the slice-ribbon conjecture is known.

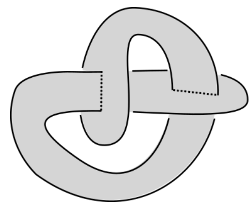

Figure 1.2: A ribbon disc for the square knot.

While it is known that there are 2-bridge knots with differing smooth and topological 4-genera [11], there are no 2-bridge knots known to be topologically yet not smoothly slice. However, Lisca's work implies that there are non smoothly slice knots whose double branched cover Casson-Gordon signature sliceness obstruction vanishes. In Chapter 3,[3] we use twisted Alexander polynomials associated to the double cover to obstruct topological sliceness for certain knots where Casson-Gordon signatures fail to provide an obstruction, such as the knot $K_{225,94}$.

Theorem A (Miller [38])**.** *There are 2-bridge knots with vanishing double branched cover Casson-Gordon signature sliceness obstruction which are not topologically slice.*

This gives additional evidence for the following conjecture.

Conjecture 1.1 ([4], [10],[38])**.** *Let K be a 2-bridge knot. Then K is topologically slice if and only if K is smoothly slice if and only if K is ribbon.*

As in the 2-bridge case, there are certain 'obviously' ribbon odd 3-strand pretzel knots. Work of Greene and Jabuka [18] uses smooth sliceness

[3]The results of Chapter 3 were originally published in the Mathematical Proceedings of the Cambridge Philosophical Society, Volume 164, Issue 1 in January 2018, DOI: 10.1017/S0305004117000172 and appear here by the kind permission of the publisher.

obstructions coming from Donaldson's theorem and Heegaard Floer homology to show that these examples are the only smoothly slice odd 3-pretzels. It is natural to ask to what extent these examples encompass topological sliceness as well. One can immediately observe that since $P(p,q,r)$ has trivial Alexander polynomial whenever $|pq+pr+qr| = 1$, there are many topologically but not smoothly slice odd 3-pretzels. That is, we have $\{\text{smoothly slice}\} \cup \{\text{trivial Alexander polynomial}\} \subseteq \{\text{topologically slice}\}$. For arbitrary families of knots this inclusion is not equality, as shown by Hedden-Livingston-Ruberman [19]; however, in Chapter 4 we use Casson-Gordon signatures to demonstrate that equality does hold for the family of odd 3-strand pretzels.[4]

Theorem B (Miller [39])**.** *Let $K = P(p,q,r)$ be a topologically slice 3-strand pretzel knot with only odd parameters. Then, up to reordering of $p, q,$ and r, one of the following is true: $q = -p$, $(p,q,r) = \pm(1, q, -q-4)$, or $|pq+pr+qr| = 1$. In particular, either K is ribbon or $\Delta_K(t) = 1$.*

In fact, Theorem 1.2 resolves Conjecture 1.1 in the genus one case. When not all the parameters of $P(p,q,r)$ are odd, similar techniques give an almost complete characterization of topological sliceness, incidentally strengthening work of Lecuona [28] in the smooth category.

Theorem C (Miller [39])**.** *Let $K = P(p,q,r)$ be a 3-strand pretzel knot with r even. Suppose also that K is not of the form $\pm P_a = \pm(a, -a-2, -\frac{(a+1)^2}{2})$ for any $a > 0$ with $a \equiv 1, 11, 37, 47, 59 \mod 60$. Then K is topologically slice iff K is smoothly slice iff $p = -q$.*

[4]The results of Chapter 4 were originally published in Algebraic & Geometric Topology, Volume 17 (2017), DOI: 10.2140/agt.2017.17.3057 and appear here by the kind permission of the publisher.

This provides strong evidence for the following conjecture, since the knots $\pm P_a$ are expected to be not even algebraically slice.

Conjecture 1.2 (Lecuona [28], Miller [39]). *A 3-strand pretzel knot $K = P(p, q, r)$ with r even is topologically slice iff K is smoothly slice iff $p = -q$.*

The classification of the smoothly slice 3-strand pretzels in [18] is established via obstructions associated to the double branched cover, and Lecuona applied this strategy to consider even pretzel knots with arbitrarily many strands [28]. There is again a family of obviously ribbon 4-strand pretzel knots, those of the form $P(2n, -2(n \pm 1), m, -m)$ for some $n, m \in \mathbb{Z}$; Lecuona and Long independently show that if $K = P(p, q, r, s)$ is smoothly slice, then $\{p, q, r, s\} = \{2n, -(2n \pm 1), m, -m\}$ for some n and m in $\mathbb{Z}$ [28, 36]. In particular, if K is slice, then it is either ribbon or at least obtained from a ribbon knot by *mutation*, illustrated in Figure 1.3. However, double branched cover

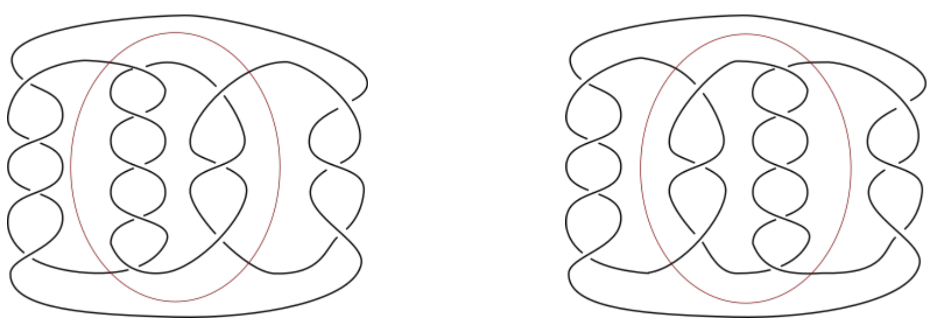

Figure 1.3: The smoothly slice pretzel knot $P(4, -5, 3, -3)$ (left) and its mutant $P(4, 3, -5, -3)$ (right), which share a double branched cover.

techniques cannot hope to distinguish mutant knots, and we are led to the following question.

Question 1.2.1. When is $K^{\pm}_{m,n} = P(2n, m, -(2n \pm 1), -m)$ slice?

In Chapter 5, we use twisted Alexander polynomials to show that many of the knots $K^{\pm}_{m,n}$ are not even topologically slice.[5] For example, the right knot of Figure 1.3 is not topologically slice.

Theorem D (Miller [37]). *Let $n \in \mathbb{N}$ and odd $m \in \mathbb{Z}$. Suppose there is a prime p dividing m such that 2 is a primitive root mod p, p does not divide $2n(2n \pm 1)$, and $n \geq \frac{p+1}{2}$. Also, assume that $(n,p) \neq (3,5)$. Then the twisted Alexander polynomials associated to the pth cyclic branched cover imply that $K^{\pm}_{m,n} = P(2n, m, -(2n \pm 1), -m)$ is not topologically slice.*

This result provides evidence for the following conjecture.

Conjecture 1.3. *A 4-strand pretzel knot $K = P(a,b,c,d)$ is topologically slice iff K is smoothly slice iff up to cyclic permutations and reversals $(a,b,c,d) = (2n, -(2n \pm 1), m, -m)$ for some $n, m \in \mathbb{Z}$.*

Finally, in Chapter 6 we address a qualitatively different question about the metric properties of satellite actions on $\mathcal{C}$. There is a metric on C coming from the minimal genus of a cobordism between two knots, and it is natural to ask how naturally occurring maps $f\colon \mathcal{C} \to \mathcal{C}$ interact with this metric. Given a pattern P in a solid torus, the classical satellite operation $K \mapsto P(K)$ descends to a map on $\mathcal{C}$. The following question is due to Cochran-Harvey [7], which they resolve in the cases $n = m$ ('always') and $|n| \neq |m|$ ('never').

Question 1.2.2. Given P a winding number m pattern and Q a winding number n pattern, when is $g_4(P(K) \# -Q(K))$ bounded independently of K?

[5] The results of Chapter 5 were originally published in the Journal of Knot Theory and its Ramifications, Volume 26, Number 7, 2017. (2017), DOI: 10.1142/S0218216517500419 and appear here by the kind permission of the publisher.

In Chapter 6, we use the Casson-Gordon signature slice genus bound of Gilmer [17] to show that the answer to Question 1.2.2 in the case $n = -m$ is ‘never’, thereby completing the following result.

Corollary 1.2.3 ([7, 40])**.** *Let P be a pattern of winding number m and Q be a pattern of winding number n. Then $g_4(P(K)\# -Q(K))$ is bounded independently of K if and only if $m = n$.*

Chapter 2

Metabelian invariants

2.1 Twisted Alexander polynomials

In general, twisted homology can be defined for spaces X which are homotopy equivalent to finite CW complexes as follows. (See [25] and [20] for a more thorough exposition.)

Let $\tilde{X}$ denote the universal cover of X, where we consider $C_*(\tilde{X})$ as acted on by the left by $\pi = \pi_1(X)$. Given M an $(\mathbb{F}[t^{\pm 1}], \mathbb{Z}[\pi])$ bimodule, where $\mathbb{F}$ is a field, the *twisted chain complex* is defined as $C_*(X, M) = M \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X})$. The twisted chain complex $C_*(X, M)$ inherits a left $\mathbb{F}[t^{\pm 1}]$-module structure from M, which descends to the *twisted homology* $H_k(X, M) = H_k(C_*(X, M))$. The kth twisted Alexander polynomials of X are defined as follows.

Definition 2.1.1. Let M be an $(\mathbb{F}[t^{\pm 1}], \mathbb{Z}[\pi])$-bimodule. The k^{th} *twisted Alexander polynomial* $\Delta^k_{X,M}(t)$ of X and M is the order of $H_k(X, M)$ as an $\mathbb{F}[t^{\pm 1}]$-module. When $k = 1$, we often call $\Delta^1_{X,M}(t) =: \Delta_{X,M}(t)$ *the twisted Alexander polynomial.*

Note that twisted Alexander polynomials are only defined up to multiplication by units in $\mathbb{F}[t^{\pm 1}]$, i.e. λt^j for $\lambda \in \mathbb{F}^\times$ and $j \in \mathbb{Z}$.

In particular, we will be interested in the twisted Alexander polynomials of prime power cyclic covers of knot exteriors, with $M = \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V$ for V

a finite dimensional $\mathbb{F}$-vector space. We will now define some notation (again following that of [20]) to be used throughout:

1. Given V a finite dimensional vector space over a field $\mathbb{F}$ and maps $\epsilon\colon \pi_1(X) \to \mathbb{Z}$ and $\phi\colon \pi_1(X) \to GL(V)$, then $M = \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V$ has a natural left $\mathbb{F}[t^{\pm 1}]$-module structure and has a right $\mathbb{Z}[\pi_1(X)]$-module structure given by $\epsilon \otimes \phi$; that is,

$$(p(t), v) \cdot \gamma = (t^{\epsilon(\gamma)} p(t), v\phi(\gamma)), \text{ for } \gamma \in \pi_1(X)$$

   We will often call the corresponding twisted Alexander polynomial $\Delta_{X,\epsilon\otimes\phi}(t)$.

2. Given X, ϵ, ϕ as above, the *reduced twisted Alexander polynomial* is $\widetilde{\Delta}_{X,\epsilon\otimes\phi}(t) = \Delta_{X,\epsilon\otimes\phi}(t)(t-1)^{-s}$, where $s = 0$ if ϕ is trivial, $s = 1$ else.

3. For K a knot, let $X(K) := S^3 - \nu(K)$ denote the exterior of K, $X_n(K)$ denote the n-fold cyclic cover of $X(K)$, and $\Sigma_n(K)$ denote the corresponding n-fold branched cover of S^3 along K. Finally, in contexts where K is clear, let $\pi = \pi_1(X(K))$ and $\pi_n = \pi_1(X_n(K))$.

4. When n is a prime power, $\Sigma_n(K)$ is a rational homology 3-sphere and there is a nondegenerate linking form

$$\mathrm{lk}\colon H_1(\Sigma_n(K)) \times H_1(\Sigma_n(K)) \to \mathbb{Q}/\mathbb{Z}.$$

   An *metabolizer* for this pairing is a subgroup M of square root order such that $\mathrm{lk}|_{M\times M}$ is identically 0. We call such a metabolizer *(linking form) invariant* if it is preserved by the action on $H_1(\Sigma_n(K))$ induced by covering transformations.

5. Let $\epsilon\colon \pi_1(X(K)) \to H_1(X(K)) \cong \mathbb{Z}$ be the Hurewicz abelianization map. Note that ϵ maps $\pi_1(X_n(K)) \subset \pi_1(X(K))$ onto $n\mathbb{Z} \subset \mathbb{Z}$, so we can define $\epsilon_n\colon \pi_n \twoheadrightarrow \mathbb{Z}$ as the composition $\epsilon_n\colon \pi_n \hookrightarrow \pi \xrightarrow{\epsilon} n\mathbb{Z} \twoheadrightarrow \mathbb{Z}$.

Definition 2.1.2. Let $\mathbb{F} \subseteq \mathbb{C}$. Define an involution of $\mathbb{F}[t^{\pm 1}]$ by

$$\bar{\ }\colon \mathbb{F}[t^{\pm 1}] \to \mathbb{F}[t^{\pm 1}],\ f(t) = \sum_{j=m}^{n} a_j t^j \mapsto \sum_{j=m}^{n} \overline{a_j} t^{-j} = \overline{f(t)}$$

A polynomial $g(t) \in \mathbb{F}[t^{\pm 1}]$ is a *norm* in $\mathbb{F}[t^{\pm 1}]$ if $g(t) = \lambda t^k f(t)\overline{f(t)}$ for some $\lambda \in \mathbb{F}^{\times}$, $k \in \mathbb{Z}$, and $f(t) \in \mathbb{F}[t^{\pm 1}]$.

We will now state the major obstruction to sliceness coming from twisted Alexander polynomials. First, observe that given any $\chi\colon H_1(X_n) \to \mathbb{Z}_d$ and ξ_d a primitive d^{th} root of unity, there is $\phi_\chi\colon \pi_n \xrightarrow{ab} H_1(X_n) \to \mathbb{Q}(\xi_d)^{\times} = GL(\mathbb{Q}(\xi_d))$ given by $\phi_\chi(\gamma) = \xi_d^{\chi(\gamma)}$. Note that, here and otherwise, we will abuse notation by using γ to refer to both an element of $\pi_1(X_n)$ and its image in $H_1(X_n)$.

In [25], the following theorem is proved by establishing a relationship between twisted Alexander polynomials of $X_n(K)$ and corresponding twisted Reidemeister torsions of $\Sigma_n(K)$, and then using duality results for Reidemeister torsion.

Theorem 2.1.3 ([25])**.** *Let K be a topologically slice knot and p, q be distinct primes, $q \neq 2$. Let $n = p^r$ and $d = q^s$ be prime powers. Then there exists an invariant metabolizer $M < H_1(\Sigma_n(K))$ such that for any $\chi\colon H_1(X_n(K)) \to \mathbb{Z}_d$ that factors through $H_1(\Sigma_n(K))$ and vanishes on M, the corresponding reduced twisted Alexander polynomial $\widetilde{\Delta}_{X_n,\epsilon_n\otimes\phi_\chi}(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ factors as a norm in $\mathbb{Q}(\xi_d)[t^{\pm 1}]$.*

However, as observed by Long in [36], the pretzel knots $K^{\pm}_{m,n}$ have only 2-torsion in their prime power cyclic branched covers. In Chapter 5 we will therefore need the following theorem, which follows immediately from the proof of Theorem 2.1.3, as observed by [35].

Theorem 2.1.4 ([25])**.** *Let K be a topologically slice knot, $p \neq 2$ prime, $n = p^r$ and $d = 2^s$. Then there exists an invariant metabolizer $M < H_1(\Sigma_n(K))$ such that for any $\chi\colon H_1(X_n(K)) \to \mathbb{Z}_d$ that factors through $H_1(\Sigma_n(K))$ and vanishes on M, the corresponding reduced twisted Alexander polynomial $\widetilde{\Delta}_{X_n,\epsilon_n\otimes\phi_\chi}(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ factors as a norm in $\mathbb{Q}(\xi_{2^m})[t^{\pm 1}]$ for some $m \in \mathbb{N}$.*

Note that the difference between the two theorems comes in whether we can assume that the reduced twisted Alexander polynomial factors as a norm over the field $\mathbb{Q}(\xi_d)$ that its coefficients naturally lie in (as in Theorem 2.1.3) or only in some larger cyclotomic extension (Theorem 2.1.4). We will be interested in Theorem 2.1.4 in the case $d = 2$, when the reduced twisted Alexander polynomial will lie in $\mathbb{Q}[t^{\pm 1}]$ and we will need to obstruct its factoring as a norm in $\mathbb{Q}(\xi_{2^m})[t^{\pm 1}]$ for any $m \in \mathbb{N}$. In fact, in Chapter 5 we will show that the reduced polynomials of interest do not even factor as norms in $\mathbb{C}[t^{\pm 1}]$, relying heavily on the fact that all coefficients are real.

In our application of Theorem 2.1.4, we will rely on the observation of [20] that when $H = H_1(\Sigma_n(K), \mathbb{Z}_d)$ is irreducible as an $\mathbb{F}_d[\mathbb{Z}_n]$-module, any invariant metabolizer $M < H_1(\Sigma_n(K))$ must have trivial image $\overline{M} < H$. So any $\chi\colon H_1(X_n(K)) \to H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ must vanish on M and if K is slice, then the reduced twisted Alexander polynomial associated to such a χ must factor as a norm. Therefore, when H is irreducible the computation of a single twisted Alexander polynomial can obstruct K's sliceness. However, when H

is not irreducible a more involved decomposition of H into irreducible components, analysis of potential metabolizers, construction of characters vanishing on said metabolizers, and computation of the corresponding twisted Alexander polynomials is required.[1]

2.1.1 Computing with Fox derivatives

First, we need the following computational result of Wada[2], who provides a way to compute twisted Alexander polynomials via Fox derivatives. Recall that given a free group $F_s = \langle x_1, \ldots, x_s \rangle$, and some $1 \leq i \leq s$ the Fox derivative with respect to x_i is the unique map $\frac{\partial}{\partial x_i} \colon F_s \to \mathbb{Z}[F_s]$ satisfying $\frac{\partial}{\partial x_j}(x_i) = \delta_{ij}$ for all $1 \leq j \leq s$, $\frac{\partial}{\partial x_k}(uv) = \frac{\partial u}{\partial x_k} + u\frac{\partial v}{\partial x_k}$ for all $u, v \in F_s$, and $\frac{\partial}{\partial x_k}(1) = 0$.

Now suppose that $\pi_1(X) = \langle x_1, \ldots, x_s : r_1, \ldots, r_t \rangle$ is a presentatation that corresponds to a handle description of X with a single 0-handle, s 1-handles, and t 2-handles. Let $\rho\colon \pi_1(X) \to GL_n(\mathbb{F})$ and $\epsilon\colon \pi_1(X) \to \mathbb{Z}$ be nontrivial homomorphisms. Let Φ be the composition

$$\Phi\colon \mathbb{Z}[\langle F_s \rangle] \twoheadrightarrow \mathbb{Z}[\pi] \xrightarrow{\epsilon\otimes\rho} M_n(\mathbb{F}[\mathbb{Z}]).$$

Then it is a classical result of Fox that the twisted homology $H_*(X, \mathbb{F}[\mathbb{Z}]^n)$ can be computed via the chain complex $\cdots \to (\mathbb{F}[\mathbb{Z}]^n)^t \xrightarrow{\delta_2} (\mathbb{F}[\mathbb{Z}]^n)^s \xrightarrow{\delta_1} \mathbb{F}[\mathbb{Z}]^n \to 0$,

[1] Example computations suggest that in the cases of interest the relevant twisted Alexander polynomials also increase significantly in complexity.

[2] Note that Wada's definition of a twisted Alexander polynomial differs from the one given above– an equivalence is proven in [25].

where

$$\delta_2 = \left[\Phi\left(\frac{\partial r_i}{\partial x_j}\right)\right]_{tn,sn} \text{ and } \delta_1 = \begin{bmatrix} \Phi(x_1 - 1) \\ \vdots \\ \Phi(x_s - 1) \end{bmatrix}.$$

Theorem 2.1.5 ([45], [25])**.** *With the setup above, there is some j such that $\Phi(x_j - 1)$ has nonzero determinant. Let $p_j \colon (\mathbb{F}[\mathbb{Z}]^n)^s \to (\mathbb{F}[\mathbb{Z}]^n)^{s-1}$ be the projection with kernel the jth copy of $\mathbb{F}[\mathbb{Z}]^n$. Define $Q_j \in \mathbb{F}[\mathbb{Z}]$ to be the greatest common divisor of the $n(s-1) \times n(s-1)$ subdeterminants of the matrix for $p_j \circ \delta_2 \colon (\mathbb{F}[\mathbb{Z}]^n)^t \to (\mathbb{F}[\mathbb{Z}]^n)^{s-1}$. If $H_1(X, \mathbb{F}^n[\mathbb{Z}])$ is torsion, then*

$$\Delta_1(X) = Q_j \frac{\Delta_0(X)}{\det(\Phi(x_j - 1))}.$$

In our case, we will have a generator x_j in $\pi_1(K)$ with $\rho(x_j) = 0$ and $\epsilon(x_j) = 1$, so $\Delta_0(X) = 1$. In addition, we will choose ρ so that for some generator x_j, we have $\det(\Phi(x_j - 1)) = 1 - t$. Finally, we will work with a reduced Wirtinger presentation, which has deficiency one and hence eliminates the need to take greatest common divisors. So we will have $\Delta_1(X) = \det \Phi(Z)(1-t)^{-1}$, where Z is obtained from $\left[\frac{\partial r_i}{\partial x_j}\right]_{s-1,s}$ by deleting the column corresponding to x_j.

2.1.2 Covers and Shapiro's lemma

We will also need a theorem of [20] that relates certain twisted Alexander polynomials of covers to those of the base space. This result, when combined with Theorem 2.1.5 will allow us to compute twisted Alexander polynomials for X_p directly from a Wirtinger presentation for $\pi_1(X)$.This will simplify the computation, even though the representations increase correspondingly in complexity, mapping elements of the fundamental group to $p \times p$ instead of 1×1 matrices.

Let p, q be distinct primes. Recall that we have $X = X(K)$ with $\pi = \pi_1(X)$, together with a canonical $\epsilon\colon \pi \to \mathbb{Z} = \langle t \rangle$ and a choice of meridian $\mu \in \pi$ with $\epsilon(\mu) = t$. The map ϵ, when combined with the obvious map $\mathbb{Z} \to \mathbb{Z}_p$ induces a p-fold cyclic cover $X_p \to X$ and corresponding surjection $\epsilon_p\colon \pi_1(X_p) \to \mathbb{Z}$. Now, suppose also that we have an irreducible $\mathbb{F}_q[\mathbb{Z}_p]$-module V, a nonzero equivariant[3] homomorphism $\rho\colon \pi_1(X_p) \to V$, and a $\mathbb{Z}_q$-vector space homomorphism $\chi\colon V \to \mathbb{Z}_q$. We would like to compute the twisted Alexander polynomial $\Delta_{X_p, \epsilon_p \otimes \rho_\chi}(t)$, where $\rho_\chi\colon = \chi \circ \rho$.

First, note that there is a group structure on $\mathbb{Z} \ltimes V$ given by $(t^i, v) \cdot (t^j, w) := (t^{i+j}, t^{-j} \cdot v + w))$, where the action of t on V is given by V's structure as an $\mathbb{F}_q[\mathbb{Z}_p]$-module. Since ρ is equivariant, there is a well-defined extension of $\epsilon|_{\pi_p} \times \rho\colon \pi_p \to p\mathbb{Z} \times V$ to a homomorphism $\tilde{\rho}\colon \pi \to \mathbb{Z} \ltimes V$ defined by $\tilde{\rho}(\gamma) = (t^{\epsilon(\gamma)}, \rho(\mu^{-\epsilon(\gamma)}\gamma))$. (In fact, [20] shows that this defines a bijection between equivariant $\rho\colon \pi_1(X_p) \to V$ and homomorphisms $\tilde{\rho}\colon \pi_1(X) \to Z \ltimes V$ with $\tilde{\rho}(\mu) = (t, 0)$.) Finally, define $\Phi\colon \pi_1(X) \to GL_p(\mathbb{Q}(\xi_q)[t^{\pm 1}])$ as the composition of $\tilde{\rho}$ with the following map $\mathbb{Z} \ltimes V \to GL_p(\mathbb{Q}(\xi_q)[t^{\pm 1}])$:

$$(t^j, v) \mapsto \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ t & 0 & \cdots & 0 \end{bmatrix}^j \begin{bmatrix} \xi_q^{\chi(v)} & 0 & \cdots & 0 \\ 0 & \xi_q^{\chi(t \cdot v)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_q^{\chi(t^{p-1} \cdot v)} \end{bmatrix}.$$

Theorem 2.1.6 ([20])**.** *Let $X, X_p, \epsilon, \rho,$ and Φ be as above, where*

- *The map $\epsilon \otimes \rho_\chi\colon \pi_1(X_p) \to GL_1(\mathbb{Q}(\xi_q)[t^{\pm 1}])$ gives $\mathbb{Q}(\xi_q)[t^{\pm 1}]$ a $(\mathbb{Q}(\xi_q)[t^{\pm 1}]$, $\mathbb{Z}[\pi_1(X_p)])$- bimodule structure*

[3] i.e. $\rho(\mu\gamma\mu^{-1}) = t \cdot \rho(\gamma)$ for any $\gamma \in \pi_1(X_p)$ and μ our preferred meridian.

- *The map* $\Phi\colon \pi_1(X) \to GL_p(\mathbb{Q}(\xi_q)[t^{\pm 1}])$ *gives* $(\mathbb{Q}(\xi_q)[t^{\pm 1}])^p$ *a* $(\mathbb{Q}(\xi_q)[t^{\pm 1}],$ $\mathbb{Z}[\pi_1(X)])$*- bimodule structure.*

Then the corresponding twisted first homology groups $H_1(X_p, \mathbb{Q}(\xi_q)[t^{\pm 1}])$ *and* $H_1(X, (\mathbb{Q}(\xi_q)[t^{\pm 1}])^p)$ *are isomorphic as* $\mathbb{Q}(\xi_q)[t^{\pm 1}]$*-modules and so* $\Delta_{X_p, \epsilon \otimes \rho_\chi}(t) = \Delta_{X,\Phi}(t)$.

2.2 Casson-Gordon signature invariants

Casson and Gordon associate to a knot K and a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ the invariant $\tau(K, n, \chi) \in L_0(\mathbb{Q}(\omega)(t)) \otimes \mathbb{Q}$. Note that $L_0(\mathbb{Q}(\omega)(t))$ is the Witt group of non-singular Hermitian forms on finite-dimensional $\mathbb{Q}(\omega)(t)$-modules, where $\omega = e^{\frac{2\pi i}{d}}$. These invariants obstruct K's topological sliceness as follows.

Theorem 2.2.1 (Casson-Gordon [4])**.** *Let* K *be a topologically slice knot and* n *a prime power. Then there exists a square-root order subgroup* $M \leq H_1(\Sigma_n(K))$*, invariant under the action of the covering transformations, with the linking form of* $\Sigma_n(K)$ *vanishing on* $M \times M$ *(i.e.* M *is a* metabolizer *for the linking form) such that if* χ *is a prime-power order character with* $\chi|_M = 0$*, then* $\tau(K, n, \chi) = 0$.

While this is a powerful sliceness obstruction, $\tau(K, n, \chi)$ cannot generally be directly computed. Instead, as originated in [4], one relates the Witt class signature $\bar{\sigma}_1(\tau(K, n, \chi))$ to a simpler signature associated to any three-manifold Y and character from $H_1(Y)$ to a cyclic group.[4] We now give the definition of this signature, following [3].

[4]We also note that, as observed by [25], the *discriminant* of $\tau(K, n, \chi)$ essentially recovers the twisted Alexander polynomial of K corresponding to χ as in the previous section.

First, whenever $X_\chi \to X$ is a cyclic d-fold cover, perhaps branched, we let $\omega = e^{\frac{2\pi i}{d}}$ and define the χ-twisted homology of X to be the $\mathbb{Q}(\omega)$ vector space $H_*^\chi(X) := H_*(C_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)) \cong H_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)$.

We now let Y be a closed 3-manifold and $\chi\colon H_1(Y) \to \mathbb{Z}_d$ an onto homomorphism. The map χ induces a d-fold cyclic cover $Y_\chi \to Y$ with a canonical generator τ for the group of covering transformations. Suppose that there is some d-fold branched cyclic cover of 4-manifolds $W_\chi \to W$ with branch set a closed surface $F \subset \mathrm{int}(W)$ such that $\partial(W_\chi \to W) = r(Y_\chi \to Y)$ for some $r \in \mathbb{N}$. Suppose also that the covering transformation $\tilde{\tau}$ of W_χ that induces rotation by $\frac{2\pi}{d}$ on the fibers of the normal bundle of the pre-image of F in W_χ induces the canonical covering transformation τ on Y_χ. We can always choose either $F = \emptyset$ or $r = 1$ by bordism group considerations and an explicit description in [3], respectively, and all of our work will be in one of these cases. The action of $\tilde{\tau}$ on $H := H_2(W_\chi, \mathbb{C})$ allows us to decompose H as the direct sum of eigenspaces $H_2^k(W_\chi)$ corresponding to eigenvalues ω^k for $k = 0, \ldots, d-1$. For $k > 0$, define $\epsilon_k(W_\chi)$ to be the signature of the intersection form of W_χ when restricted to $H_2^k(W_\chi)$. Note that $\epsilon_1(W_\chi)$ can be equivalently be defined as the signature of the twisted intersection form on $H_2^\chi(W) = H_2(W_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)$.

Definition 2.2.2. With the above set up, the k^{th} Casson-Gordon signature of (Y, χ) is

$$\sigma_k(Y, \chi) = \frac{1}{r}\left(\sigma(W) - \epsilon_k(W_\chi) - \frac{2k(d-k)}{d^2}([F] \cdot [F])\right)$$

Those familiar with the definition of $\tau(K, n, \chi)$ should note that we generally have $\sigma_1(\Sigma_n(K), \chi) \neq \bar{\sigma}_1(\tau(K, n, \chi))$. However, we can bound the difference between $\sigma_1(\Sigma_n(K),\chi)$ and $\bar{\sigma}_1(\tau(K,n,\chi))$, in a straightforward extension of Theorem 3 of [4].

Theorem 2.2.3 (Casson-Gordon [4])**.** *Let $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ be an onto homomorphism. Then*

$$|\sigma_1(\Sigma_n(K),\chi) - \bar{\sigma}_1(\tau(K,n,\chi))| \leq \dim H_1^{\chi}(\Sigma_n(K)) + 1.$$

Proof. We follow the proof of Theorem 3 in [4]. Let M_n denote the n-fold cyclic cover of the 3-manifold $S^3_0(K)$ obtained by doing 0-surgery along K. For convenience we let $\Sigma_n = \Sigma_n(K)$. Note that χ determines a map $H_1(M_n) \to \mathbb{Z}_d$, which by an abuse of notation we also refer to as χ. By the usual bordism group considerations, for some $r \in \mathbb{N}$ there is a compact 4-manifold W_n with boundary $r\Sigma_n$ such that χ extends over $H_1(W_n)$. Note that M_n can be obtained from Σ_n by a single 0-framed surgery along $\widetilde{K}$, the pre-image of K under the branched covering map. Therefore rM_n bounds a 4-manifold V_n obtained by attaching r 0-framed 2-handles to W_n. Let ν denote the nullity of the twisted intersection form on $H_2^{\chi}(V_n)$. The arguments of the proof of Theorem 3 of [4] carry over verbatim to establish the following inequality:

$$|\sigma_1(M_n,\chi) - \bar{\sigma}_1(\tau(K,n,\chi))| \leq \frac{\nu}{r}.$$

Since our covers are unbranched, Definition 2.2.2 gives us that $\sigma_1(\Sigma_n,\chi) = \frac{1}{r}(\sigma(W_n) - \sigma(H_2^{\chi}(W_n)))$ and $\sigma_1(M_n,\chi) = \frac{1}{r}(\sigma(V_n) - \sigma(H_2^{\chi}(V_n)))$. By our construction of V_n from W_n, it is straightforward to verify that $\sigma(V_n) = \sigma(W_n)$ and that $H_2^{\chi}(V_n)$ has a codimension r subspace which is isometric to $H_2^{\chi}(W_n)$. Now note that by duality the intersection form on $H_2^{\chi}(V_n)$ has nullity equal to $r\dim H_1^{\chi}(\Sigma_n)$, whereas by definition the intersection form on $H_2^{\chi}(W_n)$ has nullity ν. We therefore have the following, which when combined with our previous inequality gives the desired result.

$$
\begin{aligned}
|\sigma_1(\Sigma_n,\chi)-\sigma_1(M_n,\chi)| &= \left|\frac{1}{r}[\sigma(W_n)-\sigma(H_2^\chi(W_n))]-\frac{1}{r}[\sigma(V_n)-\sigma(H_2^\chi(V_n))]\right| \\
&= \frac{1}{r}\left|\sigma(H_2^\chi(W_n))-\sigma(H_2^\chi(V_n))\right| \\
&\leq \frac{1}{r}\left[r-(\nu-r\dim H_1^\chi(\Sigma_n))\right]=\dim H_1^\chi(\Sigma_n)+1-\frac{\nu}{r}
\end{aligned}
$$

□

The following corollary will be our main obstruction to topological sliceness.

Corollary 2.2.4 ([4]). *Suppose that K is a topologically slice knot and that $n=p^r$ is a prime power. Then there exists a metabolizer M for the linking form on $H_1(\Sigma_n(K))$ such that if χ is a character of prime power order d vanishing on M, then for any $k=1,\dots d-1$*

$$|\sigma_k(\Sigma_n(K),\chi)|\leq \dim H_1^\chi(\Sigma_n(K))+1.$$

Proof. Replacing χ with a nonzero multiple of itself permutes $\{\sigma_k(\Sigma_n(K),\chi)\}_{k=1}^{d-1}$ while preserving the property of vanishing on M, so Theorems 2.2.1 and 2.2.3 combine to give the desired result. □

If the obstruction of Corollary 2.2.4 vanishes for characters from $H_1(\Sigma_2(K))$ to $\mathbb{Z}_d$, then we will refer to K as *CG-slice* at d. The following proposition is often convenient in recognizing that $\Sigma_n(K)_\chi$ is a rational homology sphere, and hence that the bound of Corollary 2.2.4 reduces to $|\sigma_1(\Sigma_n(K),\chi)|\leq 1$.

Proposition 2.2.5 (Casson-Gordon [3]). *Suppose that Y is a rational homology sphere with $H_1(Y, \mathbb{Z}_p)$ cyclic for some prime p. Then any cyclic p^n-fold cover of Y is also a rational homology sphere.*

In order to effectively apply this obstruction, we would like to be able to compute $\sigma_k(Y, \chi)$ from an arbitrary integral surgery description of Y.

Definition 2.2.6. Let K be an oriented knot, and A an embedded annulus such that $\partial A = K \sqcup -K'$ and $\mathrm{lk}(K, K') = \lambda$. An *$\lambda$-twisted a-cable* of K is any oriented link L obtained as the union of $n = n_+ + n_-$ parallel copies of K in A such that n_+ are oriented with K, n_- opposite to K, and $n_+ - n_- = a$.

Let $L = \bigcup_{i=1}^n L_i$ be an oriented link in S^3 such that surgery along L with integer framings $\{\lambda_i\}_{i=1}^n$ gives Y. We refer to the meridian of component L_i as μ_i and let $A = [a_{ij}]$ be the linking matrix of L. The following proposition is a generalization of Lemma 3.1 of [3].

Proposition 2.2.7 (Gilmer [16]). *Let Y be obtained by integer surgery on L as above and $\chi \colon H_1(Y) \to \mathbb{Z}_d$ be an onto homomorphism. Let L_χ be a satellite of L obtained by replacing each L_i by a non-empty λ_i-twisted m_i-cable of L_i, such that $\chi(\mu_i) \equiv m_i \mod d$. Then for any $0 < k < d$,*

$$\sigma_k(Y, \chi) = \sigma(A) - \sigma_{L_\chi}(\omega^k) - \frac{2k(d-k)}{d^2} \left(\sum_{i,j=1}^{n} m_i m_j a_{ij} \right).$$

In order to effectively apply Proposition 2.2.7 we will need to compute the Tristram-Levine signatures of cables of links. The techniques of colored signatures prove useful for this, as well as providing an independent means of computation for $\sigma_1(Y, \chi)$.

2.2.1 Colored signatures of colored links

A *n-colored link* is an oriented link L together with a surjective map assigning to each component of L a color in $\{1, 2, \ldots, n\}$. We let L_i denote the sublink of L consisting of i-colored components, and call each L_i a *colored component*. A *C-complex* for a colored link L consists of a union of Seifert surfaces for the colored components of L which intersect only in a prescribed way (in 'clasps'- see [6] for the precise definition).

The *colored signature* of L is a map $\sigma_L \colon (S^1)^n \to \mathbb{Z}$ that is defined via the C-complex in a way exactly analagous to the definition of the Tristram-Levine signatures in terms of a Seifert surface for a link. The colored signature shares many properties, including a 4-dimensional interpretation, with the ordinary signatures. We need the following results, due primarily to Cimasoni-Florens [6]:

Recovery of Tristram-Levine signatures: Let L be a n-component, n-colored link, and call the underlying ordinary link L'. Then for any $\omega \in S^1 - \{1\}$, $\sigma_L(\omega, \ldots, \omega) = \sigma_{L'}(\omega) + \sum_{i<j} \operatorname{lk}(L_i, L_j)$.

Additivity: Let $L' = L'_1 \cup \cdots \cup L'_m$ and $L'' = L''_{m+1} \cup \cdots \cup L''_{m+n}$ be colored links and L be the $(m+n-1)$-colored link obtained by connected summing any component of L'_m with any component of L''_{m+1}. Then $\sigma_L(\omega_1, \ldots, \omega_m, \ldots, \omega_{m+n-1}) = \sigma_L(\omega_1, \ldots, \omega_m) + \sigma_{L''}(\omega_m, \ldots, \omega_{m+n-1})$.

Behavior under reversal and mirroring: The colored signature is invariant under global reversal of orientations. Also, letting $\bar{L}$ denote the mirror of L we have $\sigma_{\bar{L}}(\omega_1, \ldots \omega_n) = -\sigma_L(\omega_1, \ldots, \omega_n)$.

Behavior at 1: (Degtyarev-Florens-Lecuona [8]) Let L be an n-colored link and L' be the $(n-1)$-colored link obtained by deleting the nth colored component of L. Then $\sigma_L(\omega_1, \ldots, \omega_{n-1}, 1) = \sigma_{L'}(\omega_1, \ldots, \omega_{n-1})$.

Hopf link computation: Let L be either Hopf link, considered as a 2-colored link. Then the colored signature function of L is identically 0.

In Chapter 4, we will also need the following consequence of Degtyarev, Florens, and Lecuona's description of the signature of a splice in [8].

Example 2.2.8. Let L be the link shown in Figure 2.1 and $\Phi(L)$ be the satel-

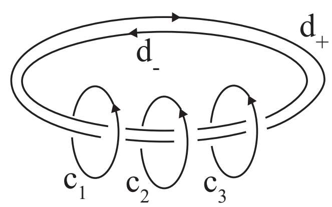

Figure 2.1: A 5-colored link L

lite of L obtained by replacing each component c_i with a coherently oriented torus link $T(a_i, p_i a_i)$ for $i = 1, 2, 3$. Observe that as an ordinary oriented link, L is isotopic to its mirror image in a way that swaps components d_+ and d_- and preserves all other components. It follows that $\sigma_L(\omega_0, \omega_0, \vec{\omega}) = 0$ for all $\omega_0 \in S^1$ and $\vec{\omega} \in (S^1)^3$. Let $\theta \in S^1$ be such that $\theta^{a_i} \neq 1$ for $i = 1, 2, 3$. Then Theorem 2.2 of [8] and the above results imply that $\sigma_{\Phi(L)}(\theta) = \sum_{i=1}^{3} \sigma_{T(a_i, p_i a_i)}(\theta)$

Finally, in some cases colored signatures give us an alternate computational method for Casson-Gordon signatures.

Theorem 2.2.9 (Cimasoni-Florens [6])**.** *Let Y be a 3-manifold obtained by surgery on a framed n-component link L with linking matrix $A = [a_{ij}]$. Let $\chi\colon H_1(Y) \to \mathbb{Z}_d$ be a character of prime-power order that takes the meridian* *of each component of L to a unit in $\mathbb{Z}_d$. Denote the lift of the image of the i^{th} meridian of L to $\{1, \ldots, d-1\}$ by m_i. Consider L as a n-colored link, and let $\omega_\chi = (\omega^{m_1}, \ldots, \omega^{m_n})$. Then*

$$\sigma_1(Y, \chi) = \sigma(A) - \left(\sigma_L(\omega_\chi) - \sum_{i<j} a_{ij}\right) - \frac{2}{d^2}\left(\sum_{i,j}(d - m_i)m_j a_{ij}\right).$$

Note that in the case that every meridian is sent to 1 and $k = 1$, Theorems 2.2.7 and 2.2.9 both reduce to the original Lemma 3.1 of [3].

Chapter 3

Topological sliceness of 2-bridge knots

In this chapter, we use twisted Alexander polynomials to show that certain algebraically slice 2-bridge knots are not topologically slice, even though all prime power Casson-Gordon signatures vanish. We also provide some computations indicating the efficacy of Casson-Gordon signatures in obstructing the smooth sliceness of 2-bridge knots. The results of this chapter originally appeared in [37] in the Mathematical Proceedings of the Cambridge Philosophical Society, Volume 164, Issue 1 (2018) and appear here by the kind permission of the publisher.

3.1 Introduction

Although 2-bridge knots $K_{p,q}$ are generally well understood, their algebraic and topological slice status is not. One of the only easily applicable statements in terms of p and q is that if $K_{p,q}$ is algebraically slice then $|H_1(\Sigma_2(K_{p,q}))| = p$ must be a square. In [4], Casson and Gordon gave the first examples of algebraically slice knots which were not ribbon, smoothly slice, or even topologically slice. For an algebraically slice knot K, every prime-power branched cover $\Sigma_{p^n}(K)$ has first homology with order equal to some square m^2. For any k dividing m and r with $0 \leq r \leq k-1$, there is a Casson-Gordon signature $\sigma_{CG}(K; p^n, k, r)$. If K is ribbon, then $\sigma_{CG}(K; p^n, k, r)$ must vanish for all choices of p^n, k, and r as above; however, sliceness (smooth or topological) only implies that these signatures must vanish for k a prime power. The signatures associated to the double branched cover of a rational knot $K_{m^2,q}$ are particularly computable; in fact, there is a combinatorial formula in terms of counts of integer points in triangles. Casson and Gordon observed in [4] that the only known rational knots for which all $\sigma_{CG}(K;2,k,r)$ vanished belonged to a certain family $\mathcal{R}$ of ribbon knots.

Conjecture 1 ([4], [10])**.** *Suppose $K_{m^2,q}$ is a 2-bridge knot. Then $K_{m^2,q}$ is ribbon if and only if all Casson-Gordon signature invariants associated to the double branched cover vanish if and only if $K_{m^2,q}$ is in $\mathcal{R}$.*

Lisca partially resolved this question by classifying the smoothly slice rational knots.

Theorem 3.1.1 ([32])**.** *$K_{p,q}$ is smoothly slice if and only if $K_{p,q}$ is ribbon if and only if $K_{p,q} \in \mathcal{R}$.*

Despite this classification, the question of exactly when the Casson-Gordon signature invariants vanish remains open.[1] Answering this question would give additional information about which 2-bridge knots are topologically slice. In particular, an affirmative answer would show that when m is a prime power the topological sliceness, smooth sliceness, and ribbonness of $K_{m^2,q}$ all coincide with the vanishing of the double branched cover Casson-Gordon signature invariants.

The first algebraically slice 2-bridge knot which is not obviously slice yet for which the Casson-Gordon signature invariants do not obstruct sliceness

[1]See [10] for more discussion of Conjecture 1 from a number-theoretic perspective.

is $K_{225,94}$, as observed in [4]. We compute a twisted Alexander polynomial associated to the double branched cover and observe that this polynomial shows that K is not topologically slice. We also give some computations indicating the effectiveness of the Casson-Gordon signature invariants (particularly when combined with the classical Alexander polynomial) at obstructing the topological sliceness of $K_{m^2,q}$ for small values of m.

3.2 Results

We have the following set-up (see Section 2.1). Let $K = K_{p,q}$ be a 2-bridge knot with Wirtinger presentation $\pi_1(X) = \langle x_1, \ldots, x_{s+1} | \, r_1, \ldots, r_s \rangle$. Suppose $p = m^2$ and let k be a prime dividing m. Let $\tilde{\rho} \colon \langle x_1, \ldots, x_{s+1} | \, r_1, \ldots, r_s \rangle \to \mathbb{Z} \ltimes \mathbb{F}_k$ be any map such that $\tilde{\rho}(x_i) = (x, v_i)$ for $i = 1, \ldots, s$, $\tilde{\rho}(x_{s+1}) = (x, 0)$, and such that whenever $x_j x_i x_j^{-1} x_l^{-1}$ is a relation then we have that $2v_j = v_i + v_l$.[2] Let $\Phi \colon \pi_1(X) \to GL_2(\mathbb{Q}(\xi_k)[t^{\pm 1}])$ be defined by

$$x_i \mapsto (x, v_i) \mapsto \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \begin{bmatrix} \xi_k^{v_i} & 0 \\ 0 & \xi_k^{-v_i} \end{bmatrix} = \begin{bmatrix} 0 & \xi_k^{-v_i} \\ t\xi_k^{v_i} & 0 \end{bmatrix},$$

and let F_Φ be the natural extension $\mathbb{Z}[\pi_1(X)] \to M_2(\mathbb{Q}(\xi_k)[t^{\pm 1}])$. If K is topologically slice, then

$$\widetilde{\Delta_K^\Phi}(t) = (t-1)^{-2} \det F_\Phi \left(\left[\frac{\partial r_i}{\partial x_j} \right]_{s,s} \right) \in \mathbb{Q}(\xi_k)[t^{\pm 1}]$$

must factor as a norm in $\mathbb{Q}(\xi_k)[t^{\pm 1}]$.

Note that the computation of $\widetilde{\Delta_K^\Phi}(t)$ as described above is easy to implement on a computer. To obstruct the topological sliceness of $K_{p,q}$ we can assume, switching (p,q) with $(p, p-q)$ if necessary, that q is even and so p/q

[2]That is, $\tilde{\rho}$ is a homomorphism of the desired form.

has an even continued fraction expansion. There is a straightforward formula for the Wirtinger presentation of $\pi_1(X(K_{p,q}))$ in terms of this even continued fraction expansion, and we obtain $\tilde{\rho}$ by solving a simple system of linear equations over $\mathbb{F}_k$. The twisted Alexander polynomial is then obtained via a simple computation; the only non-algorithmic part comes in showing that a particular $\widetilde{\Delta_K^{\Phi}}(t)$ does not factor as a norm in $\mathbb{Q}(\xi_k)[t^{\pm 1}]$.

Example 3.2.1. The knot $K = K_{225,94}$ has continued fraction expansion $[2, 2, 2, -6, -2, 2]$ and Alexander polynomial $(3t^3-6t^2+5t-1)(t^3-5t^2+6t-3)$. Work of Levine [30] implies that since the Alexander polynomial of K has no symmetric irreducible factors, the knot K is algebraically slice. One can also check that all prime-power Casson-Gordon signature invariants of the double branched cover of K vanish. However, there are Casson-Gordon signatures that obstruct K from being ribbon, and moreover Lisca's results show that K is not smoothly slice. We can show that K is not topologically slice via the computation of a single twisted Alexander polynomial, corresponding to $k = 5$. (It is perhaps interesting to note that the twisted Alexander polynomial corresponding to $k = 3$ factors as a norm even in $\mathbb{Q}[t^{\pm 1}]$.)

The reduced twisted Alexander polynomial corresponding to $k = 5$ is given by $\widetilde{\Delta_K^{\Phi}}(t) = (2 + \xi_5^2 + \xi_5^3)(t^4 + 1) - (18 + 11(\xi_5^2 + \xi_5^3))(t^3 + t) + (34 + 21(\xi_5^2 + \xi_5^3))t^2$. Note that since $\xi_5^2 + \xi_5^3 = \frac{1}{2}(-1 - \sqrt{5})$, we have that, up to multiplication by units,

$$\widetilde{\Delta_K^{\Phi}}(t) = (3 - \sqrt{5})(t^4 + 1) - (25 - 11\sqrt{5})(t^3 + t) + (47 - 21\sqrt{5})t^2.$$

To show that $K_{225,94}$ is not slice, we must obstruct this polynomial from factoring as a norm in $\mathbb{Q}(\xi_5)[t^{\pm 1}]$. Consider the Galois conjugate $g(t) = (3 + \sqrt{5})(t^4 + 1) - (25 + 11\sqrt{5})(t^3 + 1) + (47 + 21\sqrt{5})t^2$. Note that any factorization of $\widetilde{\Delta_K^{\Phi}}(t)$ in $\mathbb{Q}(\xi_5)[t^{\pm 1}]$ induces a corresponding factorization of $g(t)$, so it suffices to show that $g(t)$ is not a norm over $\mathbb{Q}(\xi_5)$. In fact, $g(t)$ has four distinct real roots and so it is enough to obstruct $g(t)$ from factoring as a norm over $\mathbb{Q}(\xi_5) \cap \mathbb{R} = \mathbb{Q}(\sqrt{5})$. So suppose that there are $\lambda, a, b, c \in \mathbb{Q}(\sqrt{5})$ such that $g(t) = \lambda(at^2 + bt + c)(ct^2 + bt + a)$; that is, such that $\lambda ac = 3 + \sqrt{5}$, $\lambda(a + c)b = -25 - 11\sqrt{5}$, and $\lambda(a^2 + b^2 + c^2) = 47 + 21\sqrt{5}$. This reduces to solving

$$\frac{(a+c)b}{ac} = -5 - 2\sqrt{5} \text{ and } \frac{a^2 + b^2 + c^2}{ac} = 9 + 4\sqrt{5} \text{ for } a, b, c \in \mathbb{Q}(\sqrt{5}).$$

It is straightforward to check using a computer algebra system that this has no solutions.

Example 3.2.2. We say $K_{m^2,q}$ is CG-fake slice if all prime-power Casson-Gordon signature invariants associated to $\Sigma_2(K)$ vanish but K is not ribbon (or, equivalently by [32], is not smoothly slice). Table 3.2.2 gives a count, for each m, of how many $K_{m^2,q}$ are CG-fake slice (counting K and $-K$ as a single entry). We omit m which are prime powers, since our computations agree with the conjecture that in this case CG signatures exactly detect smooth sliceness. These computations were done in Sage.

Example 3.2.3. The next knot we are led to consider is $K = K_{1225,466}$. K has even continued fraction expansion $[2, 2, -2, -2, -4, 4, 2, -2]$ and Alexander polynomial $(t^4 - 6t^3 + 13t^2 - 11t + 4)(4t^4 - 11t^3 + 13t^2 - 6t + 1)$. Again, we observe that K is algebraically slice and has all prime-power CG signature invariants trivial, but is not smoothly slice. The twisted Alexander polynomial

Table 3.1: Failure of Casson-Gordon signatures and Alexander polynomials to obstruct smooth sliceness

m	Number of CG-fake slice $K_{m^2,q}$	Number with $\Delta_K(t)$ a norm
$3 \cdot 5$	2	1
$3 \cdot 7$	3	0
$3 \cdot 11$	3	0
$5 \cdot 7$	10	2
$3 \cdot 13$	5	0
$3^2 \cdot 5$	3	0
$3 \cdot 17$	5	0
$5 \cdot 11$	16	2
$3 \cdot 19$	3	0

corresponding to $k = 7$ is

$$\begin{aligned}\widetilde{\Delta_K^{\Phi}}(t) =&(8 + 4(\xi^3 + \xi^4))(t^6 + 1) - (81 + 48(\xi^3 + \xi^4) - 16(\xi^2 + \xi^5))(t^5 + t)\\ &+ (287 + 189(\xi^3 + \xi^4) - 45(\xi^2 + \xi^5) + 27(\xi + \xi^6))(t^4 + t^2)\\ &- (300 + 160(\xi^3 + \xi^4) - 188(\xi^2 + \xi^5) - 75(\xi + \xi^6))t^3.\end{aligned}$$

To show that this polynomial does not factor as a norm in $\mathbb{Q}[\xi_7]$, we use the following extension of Gauss' Lemma from Herald, Kirk, and Livingston.

Lemma 3.2.4. *[20] Let k and r be primes such that $r = nk + 1$ for some $n \in \mathbb{N}$. Let $b \in \mathbb{Z}_r$ be a nontrivial k^{th} root of 1, and let $\phi\colon \mathbb{Z}[\xi_k] \to \mathbb{Z}_r$ be the ring homomorphism sending 1 to 1 and ξ_k to b. Suppose that $p(t) \in \mathbb{Z}[\xi_k](t)$ be a degree $2m$ polynomial, such that $\phi(p(t))$ also has degree $2m$. Then if $p(t)$ is a norm in $\mathbb{Q}[\xi_k](t)$, then the image $\phi(p(t))$ must factor as the product of two degree m polynomials in $\mathbb{Z}_r[t]$.*

In this case, we take $k = 7$, $r = 29 = 4 \cdot 7 + 1$, and $b = 16 \in \mathbb{Z}_{29}$. Let $\phi\colon \mathbb{Z}[\xi_7] \to \mathbb{Z}_{29}$ be defined as above with $1 \mapsto 1$ and $\xi_7 \mapsto 16$. Then

$\phi\left(\widetilde{\Delta_K^{\Phi}}(t)\right) = 20(1+6t+t^2)(1+16t+6t^2+16t^3+t^4)$ is still degree 6 and has a $\mathbb{Z}_{29}$-irreducible degree 4 factor. So, by Lemma 3.2.4, $\widetilde{\Delta_K^{\Phi}}(t)$ is not a norm over $\mathbb{Q}[\xi_7]$ and hence K is not topologically slice.

Note that the above arguments obstructing $\widetilde{\Delta_K^{\Phi}}(t)$ from factoring as a norm in the appropriate field are quite ad hoc, and there is no reason to believe that either would necessarily be effective for a larger class of 2-bridge knots. In fact, each argument fails to work for the other example. This is emphasized even more by our computations for $K_{1225,496}$. The reduced twisted Alexander polynomial for K corresponding to a nontrivial character to $\mathbb{Z}_5$ factors as a norm. While the polynomial corresponding to a nontrivial character to $\mathbb{Z}_7$ is not obviously a norm, both of the strategies used in Examples 3.2.1 and 3.2.3 fail to obstruct such a factorization.

Chapter 4

Three strand pretzel knots

In this chapter, we give a complete characterization of the topological slice status of odd 3-strand pretzel knots, proving that an odd 3-strand pretzel knot is topologically slice if and only if either it is ribbon or has trivial Alexander polynomial. We also show that topologically slice even 3-strand pretzel knots, except perhaps for members of Lecuona's exceptional family, must be ribbon. These results follow from computations of the Casson-Gordon 3-manifold signature invariants associated to the double branched covers of these knots. The results of this chapter originally appeared in [39], in Algebraic & Geometric Topology, Volume 17 and appear here by the kind permission of the publisher.

4.1 Introduction

In the years since Fox first posed the Slice-Ribbon Conjecture (Problem 1.33 on Kirby's list [24]), its validity has been established for several families of knots. The usual strategy is to give an explicit list of ribbon knots in the family and then to provide an obstruction to the smooth sliceness of all others in the family. An early example of this is the following classification of the smoothly slice rational knots due to Lisca.

Theorem 4.1.1 (Lisca [32])**.** *A rational knot is smoothly slice iff it is ribbon* *iff it is in $\mathcal{R}$.*

Note that $\mathcal{R}$ is an explicit family of rational knots known to be ribbon at least since the work of Casson and Gordon [4]. Lisca argues that if K is not in $\mathcal{R}$, then Donaldson's diagonalization theorem obstructs $\Sigma_2(K)$ from smoothly bounding a rational homology ball and hence obstructs K from being smoothly slice.

In a similar spirit, though with entirely different methods, we give an almost complete characterization of the topological sliceness of 3-strand pretzels via the computation of Casson-Gordon signatures corresponding to the double branched cover. In particular, we have the following complete characterization of topologically slice odd 3-strand pretzel knots. (Note that we call a pretzel knot $P(p_1, \ldots, p_n)$ *odd* if all of its parameters p_i are odd and *even* if one parameter is even.)

Theorem 4.1.2 (**Main Theorem A**)**.** *Let K be an odd 3-strand pretzel knot with nontrivial Alexander polynomial. Then K is topologically slice iff K is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q-4)$ for some odd $p, q \in \mathbb{N}$, in which case it is obviously ribbon.*

By work of Freedman in [15], every knot with trivial Alexander polynomial is topologically slice. The following result, originally proved by Fintushel and Stern, illustrates that this is far from true for 3-strand pretzel knots in the smooth category.

Theorem 4.1.3 (Fintushel-Stern [13])**.** *Let K be a nontrivial odd 3-strand pretzel knot with $\Delta_K(t) = 1$. Then K is not smoothly slice.*

Theorems 4.1.2 and 4.1.3 therefore together give an alternate proof of the following complete characterization of smoothly slice 3-strand pretzel knots given by Greene and Jabuka in [18]. Their arguments, like Lisca's, are smooth in nature and rely on Donaldson's theorem along with additional obstructions coming from Heegaard Floer homology.

Theorem 4.1.4 (Greene-Jabuka [18]). *Let K be an odd 3-strand pretzel knot. Then K is smoothly slice iff it is ribbon iff K is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q-4)$ for odd $p, q \in \mathbb{N}$.*

Note that both Lisca and Greene-Jabuka actually prove stronger results that completely characterize the order of rational knots and odd 3-strand pretzel knots in the smooth concordance group. Theorem 4.1.2 has the following nice corollary.

Corollary 4.1.5. *Let K be a genus one alternating knot. Then K is topologically slice iff K is ribbon.*

Proof. Let K be a genus one alternating knot. Then by work of Stoimenow in [42], K is either an odd 3-strand pretzel knot with all parameters of the same sign (and hence has nonzero signature and is not even algebraically slice) or is rational. Therefore we may assume that K is a genus one rational knot and hence (up to reflection) corresponds to the fraction $\frac{4ab+1}{2a}$ for some $a, b > 0$ (see for example Burde and Zieschang [2, Proposition 12.26]). Note that K has determinant $4ab+1 > 1$ and hence does not have trivial Alexander polynomial. Therefore, since such knots can also be described as the 3-strand pretzel knot $P(1, 2a-1, -(2b+1))$, Theorem 4.1.2 implies that K is topologically slice if and only if it is ribbon. $\square$

We also consider the topological slice status of even 3-strand pretzel knots and are able to use Casson-Gordon signatures to prove the following theorem, where for odd $a > 0$ we define P_a to be the even 3-strand pretzel knot $P(a, -a-2, -\frac{(a+1)^2}{2})$.

Theorem 4.1.6 (**Main Theorem B**)**.** *Let K be an even 3-strand pretzel knot that is not of the form $\pm P_a$ for $a \equiv 1, 11, 37, 47, 59 \mod 60$. Then K is topologically slice iff K is of the form $P(p, q, -q)$ for some even p and odd q, in which case it is obviously ribbon.*

The family $\{\pm P_a\}$ was first considered by Lecuona in [28]. Lecuona uses techniques analagous to those of Greene-Jabuka to describe the smooth sliceness of even 3-strand pretzel knots, except for this exceptional family $\{\pm P_a\}$. In fact, Lecuona's results are much broader, essentially characterizing the smooth sliceness up to mutation of all even pretzel knots not in this exceptional family. It follows from work of Jabuka in [21] that the knots $\{\pm P_a\}$ are exactly the even 3-strand pretzel knots with trivial rational Witt class and determinant one.

Theorem 4.1.7 (Lecuona [28])**.** *Let K be an even 3-strand pretzel knot that is not of the form $\pm P_a$ for any $a \equiv 1, 11, 37, 47, 49, 59 \mod 60$. Then K is smoothly slice iff it is ribbon iff it is of the form $P(p, q, -q)$ for some even p and odd q.*

Lecuona conjectures that the (non)-existence of a Fox-Milnor factorization for the Alexander polynomial obstructs even the algebraic sliceness of the $\{\pm P_a\}$ family. When combined with Theorem 4.1.6, this would imply an affirmative answer to the following conjecture.

Conjecture 2. *Let K be an even 3-strand pretzel knot. Then K is topologically slice iff K is ribbon.*

We can conveniently summarize Theorems 4.1.2 and 4.1.6 in the following (slightly weaker) statement.

Theorem 4.1.8. *Let K be a 3-strand pretzel knot with nontrivial determinant. Then K is topologically slice iff K is ribbon.*

Note that despite our almost complete understanding of topological sliceness for 3-strand pretzel knots, it remains open whether smoothly slice equals topologically slice for rational knots. Recent work of Feller and McCoy shows that there are rational knots with distinct smooth and topological 4-genera [12].

A natural next question is the extent to which double branched cover Casson-Gordon signatures obstruct the topological sliceness of pretzel knots with more than three strands. However, several difficulties arise. First, pretzel knots with more than three strands have nontrivial mutations which often persist in concordance. (See the work of Herald, Kirk, and Livingston [20] for examples.) However, even if we are willing to consider knots only up to mutation we cannot expect a complete answer from these techniques. In particular, there exist algebraically slice odd 5-strand pretzel knots with nontrivial Alexander polynomial but trivial determinant. (For example, consider $P(7, 11, 53, -5, -19)$.) There is no reason to believe that these knots are topologically slice, but there are also no double branched cover Casson-Gordon signatures to serve as sliceness obstructions.

4.2 Casson-Gordon signatures of 3-strand pretzels

We now give the outline of the proof of Theorem 4.1.2, deferring computations to later propositions.

Proof of Theorem 4.1.2. Suppose that K is an algebraically slice odd 3-strand pretzel knot with nontrivial Alexander polynomial. We will argue that either the Casson-Gordon signatures of $\Sigma_2(K)$ obstruct K's topological sliceness or that K is in fact ribbon. Since K is algebraically slice, the ordinary signature of K vanishes and an easy computation from the standard genus one Seifert surface for K shows that $pq+qr+pr<0$ (see also [21]). Also, $|H_1(\Sigma_2(K))| = -pq-qr-pr = D^2$ for some odd $D \in \mathbb{N}$. Note that since K is a genus one algebraically slice knot with nontrivial Alexander polynomial, $D^2 \neq 1$ and hence D has prime divisors. Since $pq+pr+qr<0$, the parameters p, q, and r are not all of the same sign and so via reflection and the symmetries of 3-strand pretzel knots we can assume that $p, q > 0$ and $r < 0$.

In the following cases, the existence of a prime d that divides D and satisfies the given conditions implies that the Casson-Gordon signatures of $\Sigma_2(K)$ corresponding to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness:

1. d divides p and q but not r: Proposition 4.2.1.
2. d divides r and exactly one of p and q: Proposition 4.2.3.
3. d divides all of p,q, and r: Proposition 4.2.6
4. d divides D but none of p,q, and r; $p \not\equiv q \mod d$; and (assuming without loss of generality that $q > p$) $r \neq -(4p+q)$: Proposition 4.2.9.
5. d divides D but none of p, q, and $r = -(4p+q)$: Proposition 4.2.10.

6. d divides D but none of p,q, and r; $p \equiv q \mod d$; and $d \neq 3$: Proposition 4.2.11.

Now suppose that there is no prime satisfying any of the above. It follows that p, q, and r are relatively prime, $p \equiv q \mod 3$, and D is a power of three. We show that in this case the Casson-Gordon signatures corresponding to characters of order 3 and 9 obstruct topological sliceness in Proposition 4.2.12. $\square$

We now set up for our various computation. Note that if r equals one of $-p$ and $-q$, then there is a single band move taking K to a 2-component unlink and so K is ribbon. So we suppose $r \neq -p, -q$. We start with the surgery diagram for $\Sigma_2(K)$ given in Figure 4.1, with linking matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{bmatrix}$ and $\sigma(A) = 0$. We refer to the meridians of each component by μ_0, μ_p, μ_q, and μ_r according to their framings.

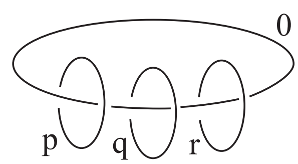

Figure 4.1: A surgery diagram L_0 for $\Sigma_2(P(p, q, r))$.

Note that A is a presentation matrix for $H_1(\Sigma_2(K))$; it is straightforward to use row and column moves and obtain the smaller presentation matrix $A' = \begin{bmatrix} p+q & p \\ p & p+r \end{bmatrix}$. Let d be any prime dividing D and suppose that d does not divide all of p, q, and r. Observe that this implies that some entry of A' is a unit in $\mathbb{Z}_d$. By choosing this entry as our pivot entry and working over $\mathbb{Z}_d$

we can use row and column moves to obtain $A'' = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$. Observe that A'' is a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$, and so we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic and hence every regular d^n-fold cyclic cover of $\Sigma_2(K)$ is a rational homology sphere (Proposition 2.2.5). In addition, when $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic any character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ will vanish on any metabolizer for the linking form. (See Lemma 8.2 of [20].) So we have the following:

Useful Fact: Suppose that $K = P(p,q,r)$ is topologically slice, d is a prime dividing $pq + qr + pr$ that does not divide all of p, q, and r, and χ is any character $H_1(\Sigma_2(K)) \to \mathbb{Z}_d$. Then $|\sigma_1(\Sigma_2(K), \chi)| \leq 1$.

4.2.1 Cases 1 and 2: d divides some but not all of p, q and r.

Proposition 4.2.1 (Case 1). *Let $K = K(p,q,r)$, where $p, q > 0$, $r < 0$, and $pq + pr + qr = -D^2$. Suppose that d is a prime that divides p and q but not r. Then the Casson-Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Proof. We start by manipulating our surgery description for $\Sigma_2(K)$. Slide the curves with framing p and q over the curve with framing r. Then convert the 0-framed 2-handle to a 1-handle and cancel the 1-handle with the r-framed 2-handle. We end with a new surgery description for $\Sigma_2(K)$ with underlying link $L = T(2, 2r)$ and framings $p + r$ and $q + r$. The linking matrix of L is $A = \begin{bmatrix} p+r & r \\ r & q+r \end{bmatrix}$ and has $\sigma(A) = 0$. Note that A considered mod d is a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$ with respect to basis $\{\mu_p, \mu_q\}$; this immediately implies that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d$, with generator $\mu_p = -\mu_q$.

By our useful fact, it suffices to show that for some $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ we have that $|\sigma_1(\Sigma_2(K), \chi)| > 1$. Define χ on $H_1(\Sigma_2(K))$ by $\chi(\mu_p) = \chi(-\mu_q) =$

1. So L_χ is the torus link $T(2,2r)$ with strands oppositely oriented. Note that $\sigma_{L_\chi}(\omega^k) = -1$ for $0 < k < d$ and so we have by Proposition 2.2.7 that

$$\sigma_k(\Sigma_2(K),\chi) = 1-2((p+r)-2r+(q+r))\frac{k(d-k)}{d^2} = 1-2\left(\frac{p+q}{d}\right)\left(\frac{k(d-k)}{d}\right)$$

Note that d divides p and q, so $p+q \geq 2d$. Note that $k(d-k) \geq (d-1)$ for all choices of $k = 1,\ldots,d-1$. Since $d \geq 3$, we have

$$|\sigma_k(\Sigma_2(K),\chi)| \geq 2\cdot 2\cdot\left(1-\frac{1}{3}\right)-1 = \frac{8}{3}-1 > 1. \qquad \square$$

Note that the above proof shows that $\sigma_k(\Sigma_2(K),\chi) < -1$ for all choices of $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ and $k = 1,\ldots,d-1$, giving the following easy corollary.

Corollary 4.2.2. *For each odd prime s, let $K_s = P(p_s, q_s, r_s)$ be an odd 3-strand pretzel knot such that $p_s, q_s > 0$ are divisible by s; $r_s < 0$ is not divisible by s; and $p_sq_s + p_sr_s + q_sr_s = -s^2$. Then $\{K_s\}$ is a basis of algebraically slice knots for a $\mathbb{Z}^\infty$ subgroup of the topological concordance group.*

Note that such K_s exist; for example, we can take $K_s = \left(s^2, s^2, -\frac{s^2+1}{2}\right)$. (Note that since s is odd s^2+1 is equivalent to 2 mod 4 and so this is an odd pretzel as desired.)

Proof. Suppose that $K = \sum_{i=1}^n a_iK_{s_i}$ is topologically slice, where each a_i is nonzero. By reflecting K, we can assume without loss of generality that $a_1 > 0$. Since K is topologically slice and $H_1(\Sigma_2(K), \mathbb{Z}_{s_i})$ is nonzero, it follows from Theorem 2.2.1 that there is some nontrivial character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_{s_1}$ such that $\bar{\sigma}_1(\tau(K,2,\chi)) = 0$. Observe that

$$H_1(\Sigma_2(K)) = \bigoplus_{i=1}^n \left(H_1(\Sigma_2(K_{s_i}))^{\oplus|a_i|}\right) = \bigoplus_{i=1}^n \left(\mathbb{Z}_{s_i}[t]/\langle t+1\rangle\right)^{\oplus|a_i|}.$$

Note that χ is trivial on each $H_1(\Sigma_2(K_{s_i}))$ factor for $i \neq 1$ and that χ can be decomposed as $\chi = \oplus_{j=1}^{|a_1|} \chi_j$, where each $\chi_j \colon H_1(\Sigma_2(K_{s_1})) \to \mathbb{Z}_{s_1}$ and at least one χ_j is nontrivial. By the additivity of Casson-Gordon signatures, $\bar{\sigma}_1(\tau(K, 2, \chi)) = \sum_{j=1}^{|a_1|} \bar{\sigma}_1(\tau(K_{s_1}, 2, \chi_j))$. However, the proof of Proposition 4.2.1 shows that $\sigma_1(\Sigma_2(K_{s_1}), \chi_j) < -1$ whenever χ_j is nontrivial and that

$$|\bar{\sigma}_1(\tau(K_{s_1}, 2, \chi_j) - \sigma_1(\Sigma_2(K_{s_1}), \chi_j)| \leq 1.$$

It follows that $\bar{\sigma}_1(\tau(K, 2, \chi_j))$ is strictly negative whenever χ_j is nontrivial (and zero when χ_j is trivial) and so $\bar{\sigma}_1(\tau(K, 2, \chi)) < 0$, which is our desired contradiction. □

Now we continue to the next case.

Proposition 4.2.3 (Case 2)**.** *Let $K = K(p, q, r)$. Suppose that there exists a prime d that divides r and exactly one of p and q, but that $r \neq -p, -q$. Then the Casson-Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Proof. The argument is exactly analogous to that of the proof of Proposition 4.2.1, except that we choose k to be $\frac{d-1}{2}$; the details are left to the reader. □

4.2.2 Case 3: d divides all of p, q, and r

In this case, we have that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$ and so there may be metabolizers $M \leq H_1(\Sigma_2(K))$ with nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$. For each such metabolizer we provide a character χ to $\mathbb{Z}_d$ vanishing on M such that the corresponding Casson-Gordon signature has sufficiently large absolute value. We first determine what "sufficiently large" is in the context of Corollary 2.2.4.

Lemma 4.2.4. *Let* $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_d$. *Then* $\dim H_1^{\chi}(\Sigma_2(K))$ *is 1 if* $\chi(\mu_p), \chi(\mu_q)$*, and* $\chi(\mu_r)$ *are all nonzero and 0 otherwise.*

Proof. By slight simplifications of the Wirtinger presentation, we obtain $\pi_1(S^3 - L_0) = \langle \mu_0, \mu_p, \mu_q, \mu_r : \mu_0\mu_p = \mu_p\mu_0, \mu_0\mu_q = \mu_q\mu_0, \mu_0\mu_r = \mu_r\mu_0 \rangle$, where μ_* is any meridian of the $*$-framed curve, for $* = 0, p, q, r$. Note that the 0-framed longitudes of the surgery curves are given with respect to this generating set by $\lambda_0 = \mu_r\mu_q\mu_p$ and $\lambda_p = \lambda_q = \lambda_r = \mu_0$. Gluing in solid tori according to the surgery framings gives new relations $\lambda_0 = \mu_r\mu_q\mu_p = 1$, $\mu_p^p\lambda_p = \mu_p^p\mu_0 = 1$, $\mu_q^q\lambda_q = \mu_q^q\mu_0 = 1$, and $\mu_r^r\lambda_r = \mu_r^r\mu_0 = 1$. We therefore have the following presentation for $\pi_1(\Sigma_2(K))$, in which generators and relators correspond respectively to the 1- and 2-cells of a cell-complex structure (with a single 0-cell) on a space homotopy equivalent to $\Sigma_2(K)$.

$$\begin{aligned}\pi_1(\Sigma_2(K)) &= \left\langle \mu_0, \mu_p, \mu_q, \mu_r : \begin{array}{l} [\mu_0, \mu_p] = [\mu_0, \mu_q] = [\mu_0, \mu_r] = 1 \\ \mu_r\mu_q\mu_p = \mu_p^p\mu_0 = \mu_q^q\mu_0 = \mu_r^r\mu_0 = 1 \end{array} \right\rangle \\ &= \left\langle \mu_p, \mu_q, \mu_r : \mu_r\mu_q\mu_p = \mu_p^p\mu_q^{-q} = \mu_p^p\mu_r^{-r} = 1 \right\rangle\end{aligned}$$

Any choice of $x, y, z \in \mathbb{Z}_d$ such that $x+y+z \equiv 0 \mod d$ will define a character χ via $\mu_p \mapsto x$, $\mu_q \mapsto y$, and $\mu_r \mapsto z$. First suppose that none of x, y, and z are equivalent to 0. Then by replacing χ with a nonzero multiple, which does not change the underlying cover, we may assume that $x = 1$.

We now follow the Reidemeister-Schreier algorithm to lift these 0-, 1-, and 2-cells to obtain a 2-complex with the same fundamental group as $\Sigma_2(K)_\chi$. Note that all subscripts are considered mod d. First, lift the single 0-cell to

d 0-cells $o_1, \ldots, o_d$. Note that μ_p has d lifts $\alpha_1, \ldots, \alpha_d$, where α_i is a 1-cell from o_i to o_{i+1}; μ_q has d lifts $\beta_1, \ldots, \beta_d$, where β_i is a 1-cell from o_i to o_{i+y}; and μ_r has d lifts $\gamma_1, \ldots, \gamma_d$, where γ_i is a 1-cell from o_i to o_{i+z}. We similarly compute the attaching maps of the d lifts of each of the 2-cells. For example, the lifts of the 2-cell corresponding to the relator $\mu_r\mu_q\mu_p$ have attaching maps of the form $\gamma_i\beta_{z+i}\alpha_{y+z+i}$ for $i = 1, \ldots, d$. Now contract along $\alpha_2, \ldots, \alpha_d$ to obtain a complex with a single 0-cell, $(2d+1)$ 1-cells, and $(3d)$ 2-cells, with a corresponding presentation for $\pi_1(\Sigma_2(K)_\chi)$. Abelianizing gives a presentation for $H_1(\Sigma_2(K)_\chi)$ with generators $a, b_1, \ldots, b_d, c_1, \ldots, c_d$ and relations $a + b_1 + c_x = 0$; $b_k + c_{x+k-1} = 0$ for $k = 2, \ldots, d$; and $\frac{p}{d}a = \frac{q}{d}(b_1 + \cdots + b_d) = \frac{r}{d}(c_1 + \cdots + c_d)$. This simplifies to

$$H_1(\Sigma_2(K)_\chi) = \langle a, b_1, \ldots, b_d : \frac{p}{d}a = \frac{q}{d}(b_1 + \cdots + b_d) = -\frac{r}{d}(b_1 + \cdots + b_d + a)\rangle$$

So as a $\mathbb{Q}$-module $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ has generators $b_1, \ldots, b_d$ and single relation $(pq + pr + qr)(b_1 + \cdots + b_d) = 0$. Note that the covering transformation of $\Sigma_2(K)_\chi$ sends b_i onto b_{i+1} for $i = 1, \ldots, d-1$, so $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ is a cyclic $\mathbb{Q}[\mathbb{Z}_d]$-module with generator b_1 and relator $(pq + pr + qr)(1 + t + t^2 + \cdots + t^{d-1})b_1$. Since $1 + \xi_d + \xi_d^2 + \cdots + \xi_d^{d-1} = 0$, we have that $H_1^\chi(\Sigma_2(K)) = H_1(\Sigma_2(K)_\chi, \mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{Z}_d]} \mathbb{Q}(\xi_d) \cong \mathbb{Q}(\xi_d)$.

When one of x, y, and z is 0, an extremely similar argument shows that $\Sigma_2(K)_\chi$ is a rational homology sphere and so $\dim H_1^\chi(\Sigma_2(K)) = 0$. $\square$

By considering the linking matrix A for L_0 with its entries taken mod d, we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is generated as a $\mathbb{Z}_d$-module by the images of μ_p, μ_q and μ_r (which we continue to refer to as μ_p, μ_q, μ_r by a mild abuse of notation) and has a single relation $\mu_p + \mu_q + \mu_r = 0$. Suppose that $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ sends μ_p to a, μ_q to b, and μ_r to c, where $0 < a, b, c < d$. We must have

$\chi(\mu_0) \equiv 0$ and $a+b+c \equiv 0 \bmod d$. We will use Proposition 2.2.7 to compute $\sigma_1(\Sigma_2(K), \chi)$, letting L_χ be the distant union of $T(a,pa)$, $T(b,qb)$, and $T(c,rc)$, each with all strands coherently oriented, along with two incoherently oriented linking 0 strands parallel to λ_0, as in Figure 4.2.2.

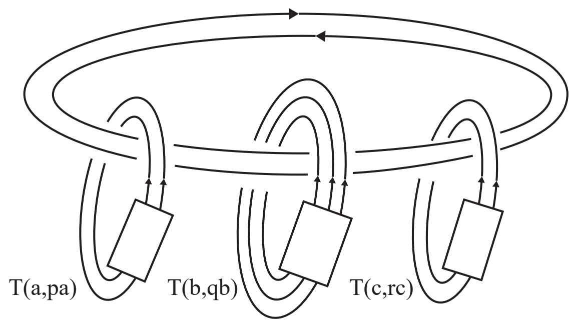

Figure 4.2: The link L_χ, pictured with $a=2, b=3, c=2$.

Note that as computed in Example 2.2.8, $\sigma_{L_\chi}(\omega) = \sigma_{T(a,pa)}(\omega) + \sigma_{T(b,qb)}(\omega) + \sigma_{T(c,rc)}(\omega)$. Also, Litherland's formula of [33] for the Tristram-Levine signature of a torus link implies that $\sigma_{T(j,jkn)}(e^{2\pi i/n}) = -2j(j-1)k$ for $0 < j < n$. While Litherland's result is stated only for torus knots, it holds for torus links as well. In particular, the underlying computation in [1] of the signature of the Brieskorn manifold $V(p,q,r)_\delta = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^p + z_2^q + z_3^r = \delta\} \cap \mathbb{D}^6$ does not depend on any relative primeness of the parameters p, q, and r.

Therefore, we have the following formula for $(*) = \sigma_1(\Sigma_2(K), \chi)$:

$$\begin{aligned}
(*) &= 0 - \sigma_{L_\chi}(\omega) - 2(a^2p + b^2q + c^2r)\left(\frac{d-1}{d^2}\right) \\
&= -\sigma_{T(a,pa)}(\omega) - \sigma_{T(b,qb)}(\omega) - \sigma_{T(c,rc)}(\omega) - 2(a^2p + b^2q + c^2r)\left(\frac{d-1}{d^2}\right) \\
&= 2a(a-1)\frac{p}{d} + 2b(b-1)\frac{q}{d} + 2c(c-1)\frac{r}{d} - 2(a^2p + b^2q + c^2r)\left(\frac{d-1}{d^2}\right) \\
&= \frac{2}{d^2}\left(a(d-a)p + b(d-b)q + c(d-c)r\right)
\end{aligned}$$

Unfortunately, we cannot conclude that $|\sigma_1(\Sigma_2(K), \chi)| > 1$ for all such choices of χ. For example, when $K = P(3{\cdot}7, 5{\cdot}7, -17{\cdot}7)$, $d = 7$, and χ sends μ_p to 2, μ_q to 4, and μ_r to 1 we have $|\sigma_1(\Sigma_2(K), \chi)| = 8/11$. However, this choice of χ does not vanish on any metabolizer for the linking form $\lambda\colon H_1(\Sigma_2(K)) \times H_1(\Sigma_2(K)) \to \mathbb{Q}/\mathbb{Z}$, so there is still some hope to obstruct K's sliceness via double branched cover Casson-Gordon signatures.

Lemma 4.2.5. *Suppose M is a metabolizer for the linking form on $H_1(\Sigma_2(K))$ with nonzero image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$. If $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ vanishes on M and takes μ_p, μ_q, μ_r to nonzero elements of $\mathbb{Z}_d$, then $\sigma_1(\Sigma_2(K), \chi)$ is an integer that is divisible by 4.*

Proof. For convenience, we write $p = dp', q = dq', r = dr'$. Note that our assumption that M has nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$ implies that there is $\alpha = x\mu_p + y\mu_q \in M$ such that not both of x and y are equivalent to 0 mod d.

The linking form is given with respect to our $\mu_0, \mu_p, \mu_q, \mu_r$ generating set for $H_1(\Sigma_2(K))$ by $-A^{-1}$ (Gordon-Litherland). Direct computation shows that $\lambda(x\mu_p + y\mu_q, x\mu_p + y\mu_q) = \frac{1}{D^2}((q+r)x^2 - 2rxy + (p+r)y^2)$. Since $\alpha \in M$, we know that D^2 and hence d^2 divides $(q+r)x^2 - 2rxy + (p+r)y^2$, and so we have Equation $(*)$: $(q'+r')x^2 - 2r'xy + (p'+r')y^2 \equiv 0 \mod d$.

Now, let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ be a character vanishing on M. As usual, we write $a = \chi(\mu_p), b = \chi(\mu_q), c = \chi(\mu_r)$, with $a+b+c \equiv 0 \mod d$. Since $\chi(\alpha) = ax + by \equiv 0 \mod d$, we can write $y = -a\bar{b}x$ and conclude that neither x nor y is equivalent to 0 mod d. Substituting into $(*)$, we obtain

$$\begin{aligned} 0 &\equiv (q'+r')x^2 - 2r'xy + (p'+r')y^2 \\ &\equiv (q'+r')x^2 + 2r'a\bar{b}x^2 + (p'+r')a^2\bar{b}^2x^2 \\ &\equiv \left[a^2\bar{b}^2p' + q' + (a\bar{b}+1)^2r'\right]x^2 \mod d. \end{aligned}$$

Multiplying through by (b^2/x^2) and recalling that $c^2 \equiv (a+b)^2 \mod d$ gives us that $a^2p' + b^2q' + c^2r' \equiv 0 \mod d$. Finally, we can write

$$\begin{aligned} \frac{d^2}{2}\sigma_1(\Sigma_2(K),\chi) &= a(d-a)p + b(d-b)q + c(d-c)r \\ &= d(a(d-a)p' + b(d-b)q' + c(d-c)r') \\ &= d^2(p'+q'+r') - d(a^2p' + b^2q' + c^2r'). \end{aligned}$$

Observe that the right side is divisible by d^2 and hence $\sigma_1(\Sigma_2(K))$ is an integer. Also, since d is odd, $a(d-a)p + b(d-b)q + c(d-c)r$ is even for any choice of a, b, and c and $\sigma_1(\Sigma_2(K),\chi)$ is divisible by 4. □

Proposition 4.2.6 (Case 3)**.** *Let $K = P(p,q,r)$, with $p, q \neq -r$ and suppose that d is a prime dividing all of p, q, and r. Then the Casson-Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Proof. Suppose that K is CG-slice at d, for a contradiction. So there exists a metabolizer $M \leq H_1(\Sigma_2(K))$ such that any character χ_0 of prime power order that vanishes on M has $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq \dim H_1^{\chi}(\Sigma_2(K)) + 1$ for all

$0 < k < d$. If there exists χ to $\mathbb{Z}_d$ vanishing on M that takes any of μ_p, μ_q, and μ_r to 0, then $\Sigma_2(K)_\chi$ is a rational homology sphere and arguments as in Cases 1 and 2 show that there is some k such that $|\sigma_1(\Sigma_2(K), k\chi)| > 1$.

So we can now assume that no such χ exists. In particular, this implies that the image of M in $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is nontrivial. So let $\chi_0 \colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ be a nontrivial character vanishing on M and taking none of μ_p, μ_q, and μ_r to 0. Since K is CG-slice, Corollary 2.2.4 and Lemma 4.2.4 combine to give us that $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq 2$ for all k. Lemma 4.2.5 gives us that $\sigma_1(K, k\chi_0)$ is an integer divisible by 4 and so $\sigma_1(\Sigma_2(K), k\chi_0) = 0$.

Now, let χ be a multiple of χ_0 such that $\chi(\mu_p) = 1$, $\chi(\mu_q) = b$, and $\chi(\mu_r) = d - b - 1$. We therefore have equation eq(1):

$$0 = \frac{d^2}{2}\sigma_1(K, \chi) = (d-1)p + b(d-b)q + (b+1)(d-b-1)r. \tag{4.1}$$

We split into cases depending on the value of b.

Case 1: $0 < b < \frac{d-1}{2}$:

Therefore $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b$, and $(2\chi)(\mu_r) = d - 2b - 2$, so we have the following.

$$0 = \frac{d^2}{2}(\sigma_1(K, 2\chi)) = 2(d-2)p + 2b(d-2b)q + (2b+2)(d-2b-2)r. \tag{4.2}$$

We then have that

$$\begin{aligned} \frac{1}{2}(2\,\mathrm{eq}(1) - \mathrm{eq}(2)) &= p + b^2q + (b+1)^2r = 0 \\ \frac{1}{2d}(4\,\mathrm{eq}(1) - \mathrm{eq}(2)) &= p + bq + (b+1)r = 0 \end{aligned}$$

It follows that $(b+1)r = -(b-1)q$ and finally that $p + q = 0$, which is our desired contradiction.

Case 2: $b = \frac{d-1}{2}$:

In this case eq(1) simplifies to show that $q+r = -\frac{4p}{d+1}$. Also, $(2\chi)(\mu_p) = 2$ and $(2\chi)(\mu_q) = (2\chi)(\mu_r) = d-1$, so we have the following.

$$0 = 2(d-2)p + (d-1)q + (d-1)r \tag{4.3}$$

Substituting our expression for $q+r$ into eq(3), we obtain that $(d^2-3d)p = 0$ and so that $d = 3$. But this implies that $q+r = -p$ and hence that p is even, which is our desired contradiction.

Case 3: $d/2 < b < d$:

Therefore $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b-d$, and $(2\chi)(\mu_r) = 2d-2b-2$. So we have the following.

$$0 = \frac{d^2}{2}(\sigma_1(K, 2\chi)) = 2(d-2)p + (2b-d)(2d-2b)q + (2b-d+2)(2d-2b-2)r \tag{4.4}$$

We then have that

$$\frac{1}{2}(2\,\mathrm{eq}(1) - \mathrm{eq}(4)) = p + (d-b)^2 q + (d-b-1)^2 r = 0$$
$$\frac{1}{2d}(4\,\mathrm{eq}(1) - \mathrm{eq}(4)) = p + (d-b)q + (d-b-1)r = 0$$

It follows that $(d-b)q = -(d-b-2)r$ and finally that $p+r = 0$, which is our desired contradiction. $\square$

4.2.3 Cases 4,5, and 6: d divides $pq+pr+qr$ but not any of p,q,r

The link L_0 considered as a 4-colored link has identically 0 colored signature, since it is a connected sum of 2-colored Hopf links. Note that since d divides none of p,q, and r, every nontrivial character χ to $\mathbb{Z}_d$ has all of $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$, and $\chi(\mu_0)$ nonzero. Theorem 2.2.9 therefore applies and we have the following simple formula for $\sigma_1(\Sigma_2(K), \chi)$.

Proposition 4.2.7. *Let $K = P(p,q,r)$ and suppose $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ has $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$, and $\chi(\mu_0)$ all nonzero. Let a, b, c, and ϵ be the unique lifts of $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$, and $\chi(\mu_0)$ to $\{1, \ldots, d-1\}$. Then*

$$\sigma_1(\Sigma_2(K), \chi) = 3 - \frac{2}{d^2} f(\chi),$$

where $f(\chi) := (d-\epsilon)(a+b+c)+(d-a)(ap+\epsilon)+(d-b)(bq+\epsilon)+(d-c)(cr+\epsilon)$.

Remark 1. Note that the parity of $a+b+c$ and of ϵ together determine the parity of $f(\chi)$; in particular, when $a+b+c$ is odd ϵ and $f(\chi)$ have opposite parities. Also, when $a+b+c = d$ we have that

$$f(\chi) = d^2 + d\epsilon + a(d-a)p + b(d-b)q + (a+b)(d-(a+b))r$$

Lemma 4.2.8. *Let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$, where d divides none of p, q, r. Then $f(\chi)$ is divisible by d^2.*

Proof. First, recall that $H_1(\Sigma_2(K))$ is presented by linking matrix A, and so our a, b, c, ϵ values must satisfy

$$a+b+c \equiv ap+\epsilon \equiv bq+\epsilon \equiv cr+\epsilon \equiv 0 \mod d.$$

We can rewrite $f(\chi)$ as

$$\begin{aligned} f(\chi) = d\,&[(a+b+c)+(ap+\epsilon)+(bq+\epsilon)+(cr+\epsilon)] \\ &- [\epsilon(a+b+c)+a(ap+\epsilon)+b(bq+\epsilon)+c(cr+\epsilon)]\,. \end{aligned}$$

The first term can immediately be seen to be divisible by d^2, so it suffices to show that $g(\chi) = \epsilon(a+b+c)+a(ap+\epsilon)+b(bq+\epsilon)+c(cr+\epsilon)$ is also divisible by d^2. Writing $ap+\epsilon = k_1 d$, $bq+\epsilon = k_2 d$, and $cr+\epsilon = k_3 d$ for $k_1, k_2, k_3 \in \mathbb{Z}$, we have

$$\begin{aligned} g(\chi) &= a(ap+\epsilon+\epsilon) + b(bq+\epsilon+\epsilon) + c(cr+\epsilon+\epsilon) \\ &= \frac{k_1 d - \epsilon}{p}(k_1 d + \epsilon) + \frac{k_2 d - \epsilon}{q}(k_2 d + \epsilon) + \frac{k_3 d - \epsilon}{r}(k_3 d + \epsilon) \\ &= \frac{k_1^2 d^2 - \epsilon^2}{p} + \frac{k_2^2 d^2 - \epsilon^2}{q} + \frac{k_3^2 d^2 - \epsilon^2}{r} \end{aligned}$$

Note that since d is relatively prime to all of p, q, and r, we can multiply through by pqr without changing the divisibility of $g(\chi)$ by d^2. We therefore have the desired result, since

$$\begin{aligned} g(\chi)pqr &= (k_1^2 d^2 - \epsilon^2)qr + (k_2^2 d^2 - \epsilon^2)pr + (k_3^2 d^2 - \epsilon^2)pq \\ &= d^2(k_1^2 qr + k_2^2 qr + k_3^2 pr) - (pq+qr+pr)\epsilon^2. \end{aligned}$$ □

Proposition 4.2.9 (Case 4)**.** *Let $K = P(p,q,r)$ with p, q, and r odd, $q \geq p > 0$, and $r < 0$ and let d be some prime dividing $pq+pr+qr$ which divides none of p, q and r. Suppose also that $r \neq -(4p+q)$ and that $p \not\equiv q$ mod d. Then the Casson-Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Proof. Assume for the sake of contradiction that K is CG-slice at d. Since $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic, for any $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ we must have

$$|\sigma_1(\Sigma_2(K), \chi)| = \left|3 - \frac{2}{d^2} f(\chi)\right| \leq 1.$$

Note that the first equality comes from Proposition 4.2.7 in the above equation. Therefore, by Lemma 4.2.8 we have $f(\chi) = d^2$ or $2d^2$.

We will work with two characters. Note that our formula for $f(\chi)$ uses the unique integer lifts of $\chi(\mu_i)$ to $\{1, \ldots, d-1\}$, so we will be careful to only write $\chi(\mu_i) = x$ if $0 < x < d$. We define χ_1 to have $\chi_1(\mu_r) = 1$, and $\chi_2 = 2\chi_1$. It follows that $\chi_1(\mu_0)$ is the unique integer ϵ in $(0,d)$ such that $\epsilon + r \equiv 0 \mod d$, $\chi_1(\mu_p)$ is the unique integer a in $(0,d)$ such that $\epsilon + ap \equiv 0 \mod d$, and $\chi_1(\mu_q) = d - a - 1$. Note that $\chi_i(\mu_p) + \chi_i(\mu_q) + \chi_i(\mu_r) = d$, so by Remark 1 $f(\chi_i)$ has the opposite parity as $\chi_i(\mu_0)$ for $i = 1, 2$. We now define some convenient notation:

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]_y = \begin{cases} x_1 \text{ if } 0 < y < \frac{d}{2} \\ x_2 \text{ if } \frac{d}{2} < y < d \end{cases} \quad \text{and} \quad \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]_{p(y)} = \begin{cases} x_1 \text{ if } y \text{ is even} \\ x_2 \text{ if } y \text{ is odd} \end{cases}.$$

We therefore have $\chi_2(\mu_p) = \left[\begin{array}{c} 2a \\ 2a-d \end{array}\right]_a$, $\chi_2(\mu_q) = \left[\begin{array}{c} d-2a-2 \\ 2d-2a-2 \end{array}\right]_a$, $\chi_2(\mu_0) = \left[\begin{array}{c} 2\epsilon \\ 2\epsilon - d \end{array}\right]_\epsilon$, $f(\chi_1) = \left[\begin{array}{c} d^2 \\ 2d^2 \end{array}\right]_{p(\epsilon)}$, and $f(\chi_2) = \left[\begin{array}{c} d^2 \\ 2d^2 \end{array}\right]_\epsilon$. (Note that if $a = \frac{d-1}{2}$, then χ_1 sends both μ_p and μ_q to $\frac{d-1}{2}$. But this implies that $p \equiv q \mod d$, which we have assumed is not the case.)

We therefore have the following two equations coming from our formulas for $f(\chi_1)$ and $f(\chi_2)$:

$$\left[\begin{array}{c} 0 \\ d^2 \end{array}\right]_{p(\epsilon)} = d\epsilon + a(d-a)p + (a+1)(d-a-1)q + (d-1)r \tag{4.5}$$

$$\left[\begin{array}{c} 0 \\ d^2 \end{array}\right]_\epsilon = d\epsilon + \left[\begin{array}{c} a(d-2a)p + (a+1)(d-2a-2)q \\ (2a-d)(d-a)p + (2+2a-d)(d-a-1)q \end{array}\right]_a + (d-2)r \tag{4.6}$$

Consider eq(7) = eq(5) − eq(6) and eq(7) = $\frac{1}{d}$ (2 eq(5) − eq(6)):

$$\left[\begin{array}{c} 0 \\ d^2 \end{array}\right]_{p(\epsilon)} - \left[\begin{array}{c} 0 \\ d^2 \end{array}\right]_\epsilon = \left[\begin{array}{c} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{array}\right]_a + r \tag{4.7}$$

$$\left[\begin{array}{c} 0 \\ 2d \end{array}\right]_{p(\epsilon)} - \left[\begin{array}{c} 0 \\ d \end{array}\right]_\epsilon = \epsilon + \left[\begin{array}{c} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{array}\right]_a + r \tag{4.8}$$

Note that the left side of eq(8) is even exactly when $\epsilon < d/2$, while the right side has the same parity as ϵ. So we can assume $\epsilon < d/2$ if and only if ϵ is even and eq(7) and eq(8) simplify to the following:

$$0 = \begin{bmatrix} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{bmatrix}_a + r \tag{4.9}$$

$$\begin{bmatrix} 0 \\ d \end{bmatrix}_\epsilon = \epsilon + \begin{bmatrix} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{bmatrix}_a + r \tag{4.10}$$

We can use eq(9) to see that if $a < d/2$, then $D = ap + (a+1)q$ and if $a > d/2$, then $D = (d-a)p + (d-a-1)q$. We will now split into cases and show that each leads to a contradiction by using eq(9) to write r in terms of a, d, p, q and substituting this expression into eq(10). Note that since d divides D, we certainly have that $d \leq D$.

Case 1: $a, \epsilon < d/2$.

By combining eq(9) and eq(10) in this case, we see that $\epsilon = a^2(p+q) + a(q-p)$ and so that

$$2a^2(p+q) < 2a^2(p+q) + 2a(q-p) = 2\epsilon < d \leq D = ap + (a+1)q.$$

It follows that $(2a^2 - a)p + (2a^2 - a - 1)q < 0$, which gives the desired contradiction.

Case 2: $a < d/2 < \epsilon$.

$$0 < d - \epsilon = -a(a-1)p - a(a+1)q < 0$$

Case 3: $\epsilon < d/2 < a$.

First, suppose that $a = d - 2$. Then eq(9) implies that $r = -(4p+q)$, which we have assumed is not the case. So we can assume that $a < d - 2$ and so

$$D = (d-a)p + (d-a-1)q < (d-a)(d-a-1)p + (d-a-1)(d-a-2)q = \epsilon < d.$$

Case 4: $d/2 < a, \epsilon$.

As in Case 3, we can assume that $a < d-2$ and so

$$0 < d - \epsilon = -(d-a)(d-a-1)p - (d-a-1)(d-a-2)q < 0. \qquad \square$$

Proposition 4.2.10 (Case 5)**.** *Suppose $K = P(p,q,r)$ for $r = -(4p+q)$. Suppose d is a prime that divides $pq+pr+qr$ but none of p,q, and r. Then either $K = P(1,q,-(q+4))$, in which case K is ribbon, or the Casson-Gordon signatures of $\Sigma_2(K)$ corresponding to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Note that $K = P(1,q,-(q+4))$ is a 2-bridge knot. If we write $q = 2k+1$, then K is a generalized twist knot corresponding to the fraction $-\frac{4(k+1)(k+2)+1}{2(k+1)}$ and has been known to be ribbon at least since [3].

Proof. Let χ be the character sending μ_p to $d-2$, μ_q and μ_r to 1, and μ_0 to ϵ. Then $\chi' = \frac{d-1}{2}\chi$ sends μ_p to 1, μ_q and μ_r to $\frac{d-1}{2}$, and μ_0 to ϵ'. Arguments as in the proof of Proposition 4.2.9 show that if $p > 1$, then at least one of $|\sigma_1(\Sigma_2(K),\chi)|$ and $|\sigma_1(\Sigma_2(K),\chi')|$ is strictly larger than 1 and hence that K is not CG slice at d. $\square$

Proposition 4.2.11 (Case 6)**.** *Suppose that d divides $pq+pr+qr$ but none of p, q, and r, that $p \equiv q \mod d$, and that $d \neq 3$. Then the Casson-Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct K's topological sliceness.*

Proof. For $i = 1, 2$, consider the characters $\chi_i : H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ defined by $\chi_i(\mu_p) = \chi_i(\mu_q) = i$, $\chi_i(\mu_r) = d - 2i$, and $\chi_i(\mu_0) = \epsilon_i$. (Note that since $d \neq 3$ we have that $d - 2i > 0$ for $i = 1, 2$.) Arguments as in the proof of

Proposition 4.2.9 show that at least one of $|\sigma_1(\Sigma_2(K), \chi_i)|$ is strictly larger than 1 and hence that K is not CG-slice at d. $\square$

Proposition 4.2.12. *Suppose that $K = P(p, q, r)$ has p, q, and r relatively prime, $|H_1(\Sigma_2(K)| = |pq + pr + qr| = 3^{2n}$ for some $n \in \mathbb{N}$, and $p \equiv q \mod 3$. Then either K is ribbon or the Casson-Gordon signatures associated to characters of order 3 and 9 obstruct K's topological sliceness.*

Proof. First, suppose that $n \geq 2$. Since p, q, and r are pairwise relatively prime, $H_1(\Sigma_2(K))$ is cyclic and any character to $\mathbb{Z}_{3^n}$ will vanish on the unique metabolizer for the linking form. Proposition 2.2.5 implies that the associated covers are rational homology spheres, so it suffices to find such a character χ with $|\sigma_1(\Sigma_2(K), \chi)| > 1$. The arguments of Propositions 4.2.9, 4.2.10, and 4.2.11 applied to $d = 9$ (according to whether $r = -(4p + q)$ and whether $p \equiv q \mod 9$) show that this is the case.

Now suppose that $n = 1$ and so $pq + pr + pq = -9$ and $r = -\frac{pq+9}{p+q}$. A slight variation on our usual arithmetic arguments then implies that $\sigma_1(\Sigma_2(K), \chi) < -1$ for some $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_3$ and hence that K is not CG-slice at $d = 3$. $\square$

4.3 Topological sliceness of even 3-strand pretzel knots.

We now outline the proof of our argument that all topologically slice even 3-strand pretzel knots are either ribbon or in Lecuona's family $\{\pm P_a\}$, leaving the details of arithmetic to the reader.

Theorem 4.3.1. *Let K be an even 3-strand pretzel knot. Suppose that K is topologically slice. Then, up to reflection, either $K = P(p, -p, q)$ for some*

$p, q \in \mathbb{N}$ *(and K is ribbon) or* $K = P_a = P\left(a, -a-2, -\frac{(a+1)^2}{2}\right)$ *for some* $a \equiv 1, 11, 37, 47, 59 \mod 60$.

Proof. Suppose that K is an algebraically slice even 3-strand pretzel. First, note that by Jabuka's computation of the rational Witt classes of pretzel knots, we can assume that either $K = P(p, -p, q)$ for some odd p and even q or that $K = P(-p, p \pm 2, q)$ for some odd p and even q such that $\det(K) = \pm 2q - p^2 \mp 2p = m^2 > 0$ (Theorem 1.11 of [21]). In the first case K is ribbon, so we assume that we are in the second case. By the symmetries of 3-strand pretzel knots, we can also assume that up to reflection $K = P(-p, p+2, q)$ for $p \in \mathbb{N}$. Then our condition that $\det(K) = 2q - p^2 - 2p > 0$ implies that $q > 0$ as well.

First, observe that if $\det(K) = 1$, then $q = \frac{(p+1)^2}{2}$ and up to reflection K is an element of Lecuona's family $\{P_a\}$. For $a \not\equiv 1, 11, 37, 47, 49, 59 \mod 60$, Theorem 4.5 of [28] states that K is not algebraically slice. When $a \equiv 49 \mod 60$, an argument analogous to the proof of in Theorem 4.5 of [28] shows that $\Delta_K(t)$ does not have a Fox-Milnor factorization and hence that K is not algebraically slice. (In particular, note that since $a \equiv 49 \mod 60$ we have that 5 divides $\frac{(a+1)^2}{4}$ and 3 divides $a + 2$. Working mod 5, $\Delta_{P_a}(t) \equiv \Pi_{1 \neq d | a} \Phi_d(t) \Pi_{1 \neq d | a+2} \Phi_d(t)$, where $\Phi_d(t)$ denotes the d^{th} cyclotomic polynomial. Since $\Phi_3(t)$ is symmetric, irreducible mod 5, and relatively prime to each $\Phi_d(t)$ for $d \neq 3$ dividing a or $a + 2$, the desired result follows.)

So we can assume that $\det(K) = m^2 > 1$ and in particular there is an (odd) prime d dividing $\det(K)$. Arguments as in the proof of Proposition 4.2.1 show that $\Sigma_2(K)$ has a surgery presentation with underlying link the coherently oriented torus link $-T(2, 2p)$ and linking matrix $\begin{bmatrix} 2 & -p \\ -p & q-p \end{bmatrix}$. It follows that $H_1(\Sigma_2(K))$ is cyclic and hence that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is certainly cyclic as well. It therefore suffices to show that there is a single $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ with $|\sigma_k(\Sigma_2(K), \chi)| > 1$ for some $1 \leq k < d$.

The construction of χ and computation of the corresponding Casson-Gordon signatures is extremely similar to the arguments of Section 4.2. Therefore, we only list the cases one must consider and leave the verification of the details to the interested reader. It is convenient to consider six cases, according to the values of K's parameters mod d: $-p \equiv q \equiv 0$; $p + 2 \equiv q \equiv 0$; $-p \equiv 2q \not\equiv 0$; $p + 2 \equiv 2q \not\equiv 0$; $-p \equiv p + 2 \not\equiv 0$; and $-p, p + 2$, and q are mutually distinct and nonzero. $\square$

Chapter 5

Four strand pretzels

In this chapter, we prove that many 4-strand pretzel knots of the form $K = P(2n, m, -2n \pm 1, -m)$ are not topologically slice, even though their positive mutants $P(2n, -2n \pm 1, m, -m)$ are ribbon. We use the sliceness obstruction of Kirk and Livingston [25] related to the twisted Alexander polynomials associated to prime power cyclic covers of knots. The results of this chapter originally appeared in [37] in the Journal of Knot Theory and its Ramifications, Volume 26, Number 7 and appear here by the kind permission of the publisher.

5.1 Introduction

In the previous chapter, we considered the sliceness of 3-strand pretzel knots; a natural extension is to ask about the sliceness of pretzel knots with arbitrarily many strands. There are partial results due to Lecuona [28] (in the case of a pretzel knot with arbitrarily many strands and an even parameter) and Long [36] (in the case of 4- or 5- strand pretzel knots) in the smooth category. In particular, note that pretzel knots of the form $P(2n, -2n \pm 1, m, -m)$ can easily be seen to be ribbon. The following theorem, due independently to Lecuona and Long, establishes that up to reordering of the parameters these are in fact all of the smoothly slice 4-strand pretzel knots.

Theorem 5.1.1 ([28], [36]). *Suppose the pretzel knot $P(a,b,c,d)$ is smoothly slice. Then $\{a,b,c,d\} = \{2n, -2n \pm 1, m, -m\}$ for some $m, n \in \mathbb{Z}$.*

In particular, the only 4-strand pretzel knots whose smooth slice status is still unresolved are the knots $P(2n, m, -2n \pm 1, -m)$ that are positive mutants of the ribbon knots $P(2n, -2n \pm 1, m, -m)$. However, the arguments used by Lisca, Greene-Jabuka, Lecuona, and Long in the proofs of the above theorems all rely on smooth sliceness obstructions that are associated to the double branched cover of a knot and so automatically vanish on mutants of smoothly slice knots.

The twisted Alexander polynomials associated to cyclic covers of knots are powerful tools for distinguishing knots from their mutants, even up to topological concordance, as demonstrated by Livingston et al in [26], [20], and [35]. For example, Herald, Kirk, and Livingston demonstrate in [20] that the 24 distinct oriented mutants of $P(3, 7, 9, 11, 15)$ are mutually distinct in the topological concordance group.

We use twisted Alexander polynomials to show that many 4-strand pretzel knots of the form $P(2n, m, -2n \pm 1, -m)$ are not even topologically slice, though their positive mutants $P(2n, -2n \pm 1, m, -m)$ are ribbon. Note that by considering $-K$ we can assume without loss of generality that $n > 0$.

Theorem 5.1.2. *Suppose $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ are such that m is odd and there exists a prime p dividing m such that 2 is a primitive root mod p., p does not divide $2n(2n \pm 1)$, and $n \geq \frac{p+1}{2}$. Also, assume that $(n, p) \neq (3, 5)$. Then $K^{\pm}_{m,n} = P(2n, m, -(2n \pm 1), -m)$ is not topologically slice.*

The argument proceeds very similarly in the two cases of $K^{+}_{m,n} = P(2n, m, -2n - 1, -m)$ and $K^{-}_{m,n} = P(2n, m, -2n + 1, -m)$. In the following, we focus on the first case $K_{m,n} := K^+_{m,n}$, leaving the precise statement and verification of the corresponding results for $K^-_{m,n}$ almost entirely to the reader.

In our context, our requirements that 2 is a primitive root mod p, that p divides m, and that p does not divide $2n(2n+1)$ are exactly those that establish that $H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2)$ is a nontrivial irreducible $\mathbb{F}_2[\mathbb{Z}_p]$-module (Lemma 5.2.1) and hence exactly those that allow us to obstruct the sliceness of $K_{m,n}$ by computing a single twisted Alexander polynomial. Note that our requirement that $n \geq \frac{p+1}{2}$ is not relevant to irreducibility; however, when $n < \frac{p+1}{2}$, the twisted Alexander polynomials we compute are norms even in $\mathbb{Q}[t^{\pm 1}]$.

Finally, in Section 5.3 we illustrate the difficulties in attempting to extend the arguments of Theorem 5.1.2 by demonstrating the additional work needed to establish that $P(6, 3, -5, -3)$ and $P(8, 7, -9, -7)$ are not topologically slice.

5.2 Proof of the main theorem

Theorem 5.1.2 will follow almost immediately from a series of lemmas and computations, which we now embark upon. We often write $m = 2k + 1$. Note that we can also write $m = p+2jp$ for some $j \in \mathbb{N}$, so $k = \frac{m-1}{2} = jp + \frac{p-1}{2}$.

5.2.1 Homology computation for the branched covers.

Lemma 5.2.1. *Let $p, m, n \in \mathbb{N}$ be as above. Then $H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2)$ is isomorphic to the irreducible $\mathbb{F}_2[\mathbb{Z}_p]$-module $V_p = \mathbb{F}_2[t]/\sum_{i=0}^{p-1} t^i$.*

Proof. First, observe that there is a Seifert matrix for $K_{m,n}$ given by $A_{m,n}$ as follows:

$$A_{m,n} = \begin{bmatrix} -B_{2n-1} & 0 & 0 & 0 & 0 \\ 0 & -B_{m-1}^T & 0 & 0 & -U_{m-1}^T \\ 0 & 0 & B_{2n}^T & 0 & U_{2n}^T \\ 0 & 0 & 0 & B_{m-1} & 0 \\ -U_{2n-1} & 0 & 0 & U_{m-1} & 0 \end{bmatrix}, \text{ where}$$

$$B_k = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{k,k} \quad \text{and } U_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1,k}.$$

Note that by taking the determinant of $tA_{m,n} - A_{m,n}^T$ we can observe that the Alexander polynomial of $K_{m,n}$ is

$$\Delta_{m,n}(t) = \left(\sum_{i=0}^{m-1} (-t)^i \right)^2.$$

Now reduce $tA_{m,n} - A_{m,n}^T$ over $\mathbb{F}_2$ coefficients to get a new presentation of $H_1(X_\infty(K_{m,n}), \mathbb{F}_2)$ as an $\mathbb{F}_2[\mathbb{Z}]$-module:

$$\begin{bmatrix} \sum_{i=0}^{m-1} t^i & \left(\sum_{i=0}^{2n} t^i\right)\left(\sum_{i=0}^{2n-1} t^i\right) \\ 0 & \sum_{i=0}^{m-1} t^i \end{bmatrix}.$$

Note that $H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2)$ is naturally an $\mathbb{F}_2[\mathbb{Z}_p]$-module, with the $\mathbb{Z}_p$ action coming from the covering transformation. In addition, this module is obtained by imposing the relation $\sum_{i=0}^{p-1} t^i = \frac{t^p-1}{t-1}$ on $H_1(X_\infty, \mathbb{F}_2)$. So

$H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2)$ is presented by

$$\begin{bmatrix} \sum_{i=0}^{m-1} t^i & \left(\sum_{i=0}^{2n} t^i\right)\left(\sum_{i=0}^{2n-1} t^i\right) \\ 0 & \sum_{i=0}^{m-1} t^i \\ \sum_{i=0}^{p-1} t^i & 0 \\ 0 & \sum_{i=0}^{p-1} t^i \end{bmatrix} \approx \begin{bmatrix} \sum\limits_{i=0}^{p-1} t^i & \left(\sum\limits_{i=0}^{2n} t^i\right)\left(\sum\limits_{i=0}^{2n-1} t^i\right) \\ 0 & \sum\limits_{i=0}^{p-1} t^i \end{bmatrix}$$

It is a well known fact that since 2 is a primitive root mod p, the polynomial $\sum_{i=0}^{p-1} t^i$ is irreducible in $\mathbb{F}_2[t]$. Note that p does not divide $2n+1$ or $2n$ and so $\sum_{i=0}^{p-1} t^i$ does not divide $\sum_{i=0}^{2n} t^i$ or $\sum_{i=0}^{2n-1} t^i$ and therefore is relatively prime to both of them in $\mathbb{F}_2[\mathbb{Z}]$.

Therefore, we can apply the Euclidean algorithm in $\mathbb{F}_2[\mathbb{Z}]$ via Tietze-like moves to simplify the above matrix and demonstrate that $H_1(\Sigma_p(K_m, n), \mathbb{F}_2)$ is a cyclic $\mathbb{F}_2[\mathbb{Z}]$-module and hence is a cyclic $\mathbb{F}_2[\mathbb{Z}_p]$- module as well. So $H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2) \cong \mathbb{F}_2[t]/q(t)$ for some $q(t)$ dividing $\sum_{i=0}^{p-1} t^i$. Finally, note that $q(t) \neq 1$, since we can compute from the Alexander polynomial that $H_1(\Sigma_p(K_{m,n}))$ has nontrivial 2-torsion. Therefore, since $\sum_{i=0}^{p-1} t^i$ is irreducible in $\mathbb{F}_2[\mathbb{Z}]$ we can conclude that $H_1(\Sigma_p(K_{m,n}), \mathbb{F}_2)$ is isomorphic to the irreducible $\mathbb{F}_2[\mathbb{Z}_p]$-module $V_p = \mathbb{F}_2[t]\Big/ \sum_{i=0}^{p-1} t^i$. $\square$

An identical argument shows that $H_1(\Sigma_p(K^-_{m,n}), \mathbb{F}_2) \cong V_p$ is irreducible whenever 2 is a primitive root mod p and p does not divide $2n(2n-1)$.

To apply the computational simplifications of [20] as in Lemma 2.1.6, we choose a nonzero equivariant homomorphism $\rho\colon \pi_1(X_p(K_{m,n})) \to V_p$ and extend $\epsilon \times \rho$ to $\tilde{\rho}\colon \pi(X(K_{m,n})) \to \mathbb{Z} \ltimes V_p$ with $\tilde{\rho}(\mu) = (x, 0)$, where μ is a preferred meridian in $\pi(X(K_{m,n}))$. Note that any equivariant ρ must factor through $H_1(\Sigma_p(K))$, since it satisfies $\rho(\mu^p) = \rho(\mu\mu^p\mu^{-1}) = t \cdot \rho(\mu^p)$. We will instead directly construct $\tilde{\rho}$.

The following is a Wirtinger presentation for the knot group of $K_{m,n} = P(2n, 2k+1, -2n-1, -2k-1)$, where $a \cdot b$ denotes aba^{-1} and for convenience we let $M = 4n + 4k + 3$.

$$\left\{ x_1, \cdots, x_M : \begin{array}{ll} x_{i+1} = x_{i+3n+3k+3} \cdot x_i, & 1 \le i \le k \\ x_{i+1} = x_{i+2n+2k+2} \cdot x_i, & k+1 \le i \le n+k+1 \\ x_i = x_{i+n+k+1} \cdot x_{i+1}, & n+k+2 \le i \le n+2k+2 \\ x_i = x_{i+n+2k+2} \cdot x_{i+1}, & n+2k+3 \le i \le 2n+2k+1 \\ x_{2n+2k+2} = x_1 \cdot x_{2n+2k+3} & \\ x_i = x_{i-(n+k)} \cdot x_{i+1}, & 2n+2k+3 \le i \le 2n+3k+2 \\ x_{i+1} = x_{i-(2n+2k+1)} \cdot x_i, & 2n+3k+3 \le i \le 3n+3k+2 \\ x_{i+1} = x_{i-(3n+3k+2)} \cdot x_i, & 3n+3k+3 \le i \le 3n+4k+3 \\ x_i = x_{i-(2n+2k+1)} \cdot x_{i+1} & 3n+4k+4 \le i \le M-1 \\ x_M = x_{2n+2k+2} \cdot x_1 & \end{array} \right.$$

We choose as preferred meridian $\mu = x_1$. Note that since $\tilde{\rho}$ extends some $\epsilon \times \rho$, we must have $\tilde{\rho}(x_i) = (x, v_i)$ for each of the Wirtinger generators. The Wirtinger relation $x_l = x_i \cdot x_j$ implies that $v_l = (1-t)v_i + tv_j$ in V_p. After some simple reductions of the linear relations coming from the above Wirtinger presentation, we see that $\tilde{\rho}$ is determined by our choice of $v_1, v_{k+1}, v_{n+k+2}, v_{n+2k+3}, v_{2n+2k+3}, v_{2n+3k+3}, v_{3n+3k+3},$ and $v_{3n+4k+4}$. In addition, any choice satisfying $v_1 = v_{n+2k+3} = v_{2n+2k+3} = v_{3n+4k+4}$ and $v_{k+1} = v_{n+k+2} = v_{2n+3k+3} = v_{3n+3k+3}$ determines a valid $\tilde{\rho}$.

Since we require that $\tilde{\rho}(\mu) = \tilde{\rho}(x_1) = (x, v_1) = (x, 0)$, the map $\tilde{\rho}$ is entirely determined by our choice of $a = v_{k+1}$.[1] In fact, since we will also choose $\chi\colon V_p \to \mathbb{F}_2$, there are essentially only two distinct choices of $\tilde{\rho}$: the trivial map with $a = 0$ and the map corresponding to $a = 1$. We will choose $a = 1$.

[1]Note that when n does not satisfy our divisibility requirements with regards to p, the map described above is still a homomorphism, but there are many other choices.

We will also define[2] $\chi\colon V_p \to \mathbb{F}_2$ by $\chi(t^i) = \begin{cases} 1 & \text{if } i = 0, 2 \\ 0 & \text{else} \end{cases}$ and define $\rho_\chi\colon \pi_1(X_p(K)) \to \mathbb{F}_2$ as the composition

$$\rho_\chi\colon \pi_1(X_p(K)) \xrightarrow{ab.} H_1(X_p(K)) \xrightarrow{i_*} H_1(\Sigma_p(K)) \xrightarrow{\rho} V_p \xrightarrow{\chi} \mathbb{F}_2.$$

Therefore, by Theorem 2.1.6 we have that $\Delta_{X_p(K),\epsilon\otimes\rho_\chi}(t) = \Delta_{X(K),\Phi}(t)$, where $\Phi\colon \pi_1(K) \to GL_p(\mathbb{Q}[t^{\pm 1}]$ is defined by

$$\begin{array}{r} x_{n+k+3}, \ldots, x_{2n+3k+2}, x_{3n+4k+4}, \ldots, x_{4n+4k+3}, x_1 \mapsto x \\ x_2, \ldots, x_{n+k+2}, x_{2n+3k+3}, \ldots, x_{3n+4k+3} \mapsto y \end{array}, \text{where}$$

$$x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ t & 0 & 0 & \cdots & 0 \end{bmatrix}_{p\times p} \quad y = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -t & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{p\times p}.$$

Note that an almost identical construction gives maps $\tilde{\rho}^*\colon \pi_1(K^-_{m,n}) \to \mathbb{Z} \ltimes V_p$ and $\chi^*\colon V_p \to \mathbb{F}_2$.

5.2.2 Computation of the reduced twisted Alexander polynomial

First, recall that the twisted Alexander polynomial is only well defined up to units in $\mathbb{Q}[t^{\pm 1}]$. We therefore let $\doteq$ denote equality up to multiplication by units and frequently omit factored-out powers of t.

Lemma 5.2.2. *Let $m = 2k+1, n, p \in \mathbb{N}$ be such that p divides m. Suppose that $n \geq \frac{p+1}{2}$ and that p does not divide $2n(2n+1)$. So $2n = bp + a$ for some $0 < a < p-1$ and $b \geq 1$.*[3]

[2]Note that this is a significant choice: for $p > 3$, sample computations indicate that different choices of χ give very different twisted Alexander polynomials.

Then, with ρ and χ as above, the reduced twisted Alexander polynomial for $K_{m,n}$ is given by $\tilde{\Delta}_{m,n}(t) = f_b(t)g_n(t)h_k(t)^2(t-1)^{-2}$ where $h_k(t) \in \mathbb{Z}[t]$ and

$$f_b(t) := 2\sum_{i=0}^{2b} t^i + t^b = 2t^{2b} + 2t^{2b-1} + \cdots + 2t^{b+1} + 3t^b + 2t^{b-1} + \cdots + 2,$$

$$g_n(t) := (4a-6)\sum_{i=0}^{2b}(-t)^i + (-t)^b - 4(p-4)t\left(\sum_{i=0}^{b-1}(-t)^i\right)^2.$$

As usual, an analogous result holds for $K^-_{m,n} = P(2n, m, -2n+1, -m)$, where instead of $f_b(t)$ as above we have $f^*_b(t) = 2\sum_{i=0}^{2b} t^i - t^b = 2t^{2b} + 2t^{2b-1} + \cdots + 2t^{b+1} + t^b + 2t^{b-1} + \cdots + 2$. We also have a different $g^*_n(t)$, which is a degree $2b$ polynomial defined by a formula very similar to that of $g_n(t)$.

Proof. First note that by Lemmas 2.1.6 and 2.1.5 that if we let Z be the reduced Fox derivative matrix of a reduced Wirtinger presentation for $\pi_1(X(K))$ then $\widetilde{\Delta}_{X_p(K),\epsilon\otimes\rho_\chi}(t) = \widetilde{\Delta}_{X(K),\Phi}(t) = \Delta_{X(K),\Phi}(t)(t-1)^{-1} = \det(\Phi(Z))(t-1)^{-2}$. So it suffices to show that $\det(\Phi(Z)) \doteq f_b(t)g_n(t)h_k(t)^2$ as defined above.

We will use the following simplification of our original Wirtinger presentation:[4]

$$\pi_1(K) = \left\{ \begin{array}{lcll} & & a = (\eta\alpha)^{-n}\alpha(\eta\alpha)^n & e = (\eta\alpha)^{-(n-1)}\alpha^{-1}(\eta\alpha)^n \\ a, b, c, e & \text{s.t.} & b = (\beta\gamma)^n\beta(\beta\gamma)^{-n} & c = (\beta\gamma)^{n+1}\beta^{-1}(\beta\gamma)^{-n} \\ \alpha, \beta, \gamma, \eta & & \gamma = (ec)^k e(ec)^{-k} & \eta = (ec)^{k+1}e^{-1}(ec)^{-k} \\ & & \beta = (ba)^{-k}a(ba)^k & \end{array} \right.$$

[3]Note that this computation does not depend on 2 being a primitive root mod p, though it does use the divisibility relations between $p, m,$ and n and that $n \geq \frac{p+1}{2}$. In particular, this formula does give non-norm reduced twisted Alexander polynomials for many $K_{m,n}$ not satisfying the conditions of Theorem 5.1.2– for example, for $K = P(8, 7, -9, -7)$. However, when 2 is not primitive mod p, this is not enough to obstruct sliceness for K.

[4]Note that $a = x_{2k+2n+3}$, $b = x_{k+n+2}$, $c = x_{3k+3n+3}$, $e = x_1$, $\alpha = x_{2k+n+3}$, $\beta = x_{3k+2n+3}$, $\gamma = x_{k+1}$, and $\eta = x_{4k+3n+4}$. So $\Phi(a) = \Phi(e) = \Phi(\alpha) = \Phi(\eta) = x$ and $\Phi(b) = \Phi(c) = \Phi(\beta) = \Phi(\gamma) = y$.

The Fox derivatives of these relations are given by

$$\begin{aligned}
&(\eta\alpha)^n da + \left[(1-a)\textstyle\sum_{i=0}^{n-1}(\eta\alpha)^i\eta - 1\right] d\alpha + \left[(1-\alpha)\textstyle\sum_{i=0}^{n-1}(\eta\alpha)^i\right] d\eta,\\
&\alpha(\eta\alpha)^{n-1}de + \left[(1-\eta)\textstyle\sum_{i=0}^{n-1}(\alpha\eta)^i\right] d\alpha + \left[\alpha\textstyle\sum_{i=0}^{n-2}(\eta\alpha)^i - \textstyle\sum_{i=0}^{n-1}(\eta\alpha)^i\right] d\eta,\\
&db + \left[(b-1)\textstyle\sum_{i=0}^{n}(\beta\gamma)^i\right] d\beta + \left[(b-1)\textstyle\sum_{i=0}^{n-1}(\beta\gamma)^i\beta - (\beta\gamma)^n\beta\right] d\gamma,\\
&dc + \left[(c-1)\textstyle\sum_{i=0}^{n-1}(\beta\gamma)^i - (\beta\gamma)^n\right] d\beta + \left[(c-1)\textstyle\sum_{i=0}^{n-1}(\beta\gamma)^i\beta\right] d\gamma,\\
&d\gamma + \left[(\gamma-1)\textstyle\sum_{i=0}^{k-1}(ec)^i - (ec)^k\right] de + \left[(\gamma-1)\textstyle\sum_{i=0}^{k-1}(ec)^i e\right] dc,\\
&d\eta + \left[(\eta-1)\textstyle\sum_{i=0}^{k}(ec)^i\right] de + \left[(\eta-1)\textstyle\sum_{i=0}^{k-1}(ec)^i e - (ec)^k e\right] dc, \text{ and}\\
&(ba)^k d\beta + \left[(1-a)\textstyle\sum_{i=0}^{k-1}(ba)^i\right] db + \left[-1 + (1-a)\textstyle\sum_{i=0}^{k-1}(ba)^i b\right] da.
\end{aligned}$$

So the image of the reduced Fox derivative matrix (with column corresponding to $e = \mu$ deleted) is $\Phi(Z) = [\Phi(Z)_L \;\; \Phi(Z)_R]$, where $\Phi(Z)_L$ and $\Phi(Z)_R$ are given as follows.

$$\Phi(Z)_L = \begin{bmatrix}
x^{2n} & 0 & 0\\
0 & 0 & 0\\
0 & 1 & 0\\
0 & 0 & 1\\
0 & 0 & (*)_{5,3}\\
0 & 0 & (*)_{6,3}\\
y\sum_{i=0}^{k-1}(xy)^i - \sum_{i=0}^{k}(xy)^i & (1-x)\sum_{i=0}^{k-1}(yx)^i & 0
\end{bmatrix},$$

where $(*)_{5,3} = (y-1)\displaystyle\sum_{i=0}^{k-1}(xy)^i x$ and $(*)_{6,3} = (x-1)\displaystyle\sum_{i=0}^{k-1}(xy)^i x - (xy)^k x$.

$$\Phi(Z)_R = \begin{bmatrix}
-\sum_{i=0}^{2n}(-x)^i & \sum_{i=0}^{2n-1}(-x)^i & 0 & 0\\
\sum_{i=0}^{2n-1}(-x)^i & -\sum_{i=0}^{2n-2}(-x)^i & 0 & 0\\
0 & 0 & -\sum_{i=0}^{2n+1}(-y)^i & \sum_{i=1}^{2n+1}(-y)^i\\
0 & 0 & -\sum_{i=0}^{2n}(-y)^i & \sum_{i=1}^{2n}(-y)^i\\
0 & 0 & 0 & 1\\
0 & 1 & 0 & 0\\
0 & 0 & (yx)^k & 0
\end{bmatrix}$$

The matrix $\Phi(Z) = [\Phi(Z)_L \;\; \Phi(Z)_R]$ can be shown via simple row and column moves to have the same determinant (up to units) as the matrix

$$\widehat{\Phi(Z)} = \begin{bmatrix} -A_n & 0 & B_n \\ y(xy)^k A_n & C_k & 0 \\ D_{k,n} & C_k & E_k \end{bmatrix},$$

where $A_n = -\sum_{i=0}^{2n} (-y)^i$, $B_n = \sum_{i=0}^{2n-1} (-x)^i$, $C_k = 1 + (y-1)x \sum_{i=0}^{k-1} (yx)^i$,
$D_{k,n} = \sum_{i=0}^{2n+1} (-y)^i + (y-1)x \sum_{i=0}^{k-1} (yx)^i (-y)^{2n+1}$, $E_k = 1 + (x-1)y \sum_{i=0}^{k-1} (xy)^i$.

Observe that

$$\begin{aligned} \det(\widehat{\Phi(Z)} &= \det(C_k) \det \begin{bmatrix} -A_n & 0 & B_n \\ y(xy)^k A_n & I & 0 \\ D_{k,n} & I & E_k \end{bmatrix} \\ &= \det(C_k) \det \begin{bmatrix} -A_n & B_n \\ D_{k,n} - y(xy)^k A_n & E_k \end{bmatrix} \\ &= \det(C_k) \det(E_k) \det \begin{bmatrix} -A_n & B_n \\ E_k^{-1} \left(D_{k,n} - y(xy)^k A_n\right) & I \end{bmatrix} \\ &= \det(C_k) \det(E_k) \det(-A_n - B_n E_k^{-1} (D_{k,n} - y(xy)^k A_n)). \end{aligned}$$

By Lemma 5.4.1, $\det(C_k) = \det(E_k)$. Let $h_k(t) := \det(C_k) = \det(E_k)$, so

$$\det(\widehat{\Phi(Z)}) \doteq h_k(t)^2 \det(A_n + B_n E_k^{-1} (D_{k,n} - y(xy)^k A_n)).$$

Note that the entries of C_k are in $\mathbb{Z}[t]$, so $h_k(t) \in \mathbb{Z}[t]$. By Lemma 5.4.1, we also have that the matrix $E_k^{-1}(D_{k,n} - y(xy)^k A_n))$ is independent of k. So let $k_0 := \frac{p-1}{2}$ and $F_n := E_{k_0}^{-1}(D_{k_0,n} - y(xy)^{k_0} A_n))$. Then

$$\det(\widehat{\Phi(Z)}) \doteq h_k(t)^2 \det(A_n + B_n F_n). \tag{5.1}$$

Now, recall that $2n, 2n+1 \not\equiv 0 \mod p$ and so we can write $2n = bp + a$ for $0 < a < p-1$. By Lemma 5.4.2, we have that $\det(A_n + B_n F_n) \doteq f_b(t) \det g_n(t)$, where $f_b(t)$ is as above and

$$g_n(t) := \frac{\det(G_n)}{1+t} = \det \begin{bmatrix} (p-a-2)\beta_b(t) & -1 & -1 \\ \Psi_b(t) & 2(-1)^b & -2t^{b+1} \\ (a-2)\beta_{b+1}(t) & 1 & -t \end{bmatrix} (1+t)^{-1},$$

where $\Psi_b(t) = (-1)^b t \left(2\sum_{i=0}^{2b}(-t)^i + (-t)^b\right)$ and $\beta_b(t) = 2\sum_{i=1}^{b}(-t)^i$.
Observe that

$$\begin{aligned} g_n(t)(t+1) = -\Psi_b(t)(t+1) &+ 2t(p-a-2)\beta_b(t)(t^b + (-1)^{b+1}) \\ &+ 2(a-2)\beta_{b+1}(t)(t^{b+1} + (-1)^b) \end{aligned}$$

The right side of this equation can be manipulated to give the following expression for $g_n(t)$.

$$g_n(t) = 2\sum_{i=0}^{2b}(-t)^i + (-t)^b - 4(p-4)t\left(\sum_{i=0}^{b-1}(-t)^i\right)^2 + 4(a-2)\sum_{i=0}^{2b}(-t)^i$$

and so

$$g_n(t) = (4a-6)\sum_{i=0}^{2b}(-t)^i + (-t)^b - 4(p-4)t\left(\sum_{i=0}^{b-1}(-t)^i\right)^2. \tag{5.2}$$

Therefore, combining (5.1), (5.2), and Lemma 5.4.2 we have as desired that

$$\widetilde{\Delta}_{m,n}(t) = \det(\widehat{\Phi(Z)})(t-1)^{-2} = h_k(t)^2 f_b(t) g_n(t)(t-1)^{-2}.$$

□

5.2.3 $\widetilde{\Delta}_{m,n}(t)$ is not a norm.

We will now show that $\widetilde{\Delta}_{m,n}(t)$ is not a norm in $\mathbb{C}[t^{\pm 1}]$ and hence is certainly not a norm in any $\mathbb{Q}(\xi_{2^n})[t^{\pm 1}]$.

Theorem 5.2.3. *Let $m, n, p \in \mathbb{N}$ be such that p divides m but not $2n(2n+1)$ and such that $(n, p) \neq (3, 5)$. Let $f_b(t), g_n(t)$ be as above and $h_k(t) \in \mathbb{Z}[t]$. Then $\tilde{\Delta}_{m,n}(t) = f_b(t)g_n(t)h_k(t)^2(t-1)^{-2}$ is not a norm in $\mathbb{C}[t^{\pm 1}]$.*

Proof. First, observe that our map $\rho\colon \pi_1(X_p(K_{m,n})) \to \mathbb{F}_2 \hookrightarrow \mathbb{Q}^x \cong GL_1(\mathbb{Q})$ is trivially unitary. By Corollary 5.2 of [25], the corresponding reduced twisted Alexander polynomial $\tilde{\Delta}_{m,n}(t)$ is a symmetric polynomial, up to multiplication by units in $\mathbb{Q}[t^{\pm 1}]$.[5]

Therefore, since $f_b(t)$ and $g_n(t)$ have symmetric coefficients, $h_k(t)^2$ and hence $h_k(t) \in \mathbb{Z}[t]$ must as well So

$$h_k(t)^2 = t^{\deg(h_k)} h_k(t) h_k(t^{-1}) = t^{\deg(h_k)} h_k(t) \overline{h_k(t^{-1})}$$

is a norm, as is $(t-1)^{-2}$. So it suffices to show that $f_b(t)g_n(t)$ is not a norm.

Note that both $g_n(t)$ and $f_b(t)$ are of degree $2b$ and so we can check by explicitly computing the three highest-degree coefficients of each polynomial that for $(n, p) \neq (3, 5)$, the polynomial $g_n(t)$ is not a multiple of $f_b(t)$. Therefore, our result will follow from showing that $f_b(t)$ is irreducible in $\mathbb{Q}[t]$ and not a norm in $\mathbb{C}[t]$, as is checked in Lemma 5.2.5. $\square$

[5]That is, there is some $\lambda \in \mathbb{Q}^\times$ and $k \in \mathbb{Z}$ such that $\tilde{\Delta}_{m,n}(t^{-1}) = \lambda t^k \tilde{\Delta}_{m,n}(t)$. When we say a polynomial is "symmetric", we will always mean it in this sense, up to multiplication by a unit in the appropriate polynomial ring.

We need the following result of P. Lakatos, which describes when perturbations of certain products of cyclotomic polynomials have only unit norm roots.

Theorem 5.2.4 ([27]). *Suppose that $p(z) \in \mathbb{R}[z]$ is such that there are $l, a_0, \cdots, a_{\lfloor \frac{r}{2} \rfloor} \in \mathbb{R}$ and $r \geq 2$ with*

$$p(z) = l(z^r + z^{r-1} + \cdots + z + 1) + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} a_k (z^{r-k} + z^k).$$

If $|l| \geq 2 \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} |a_k|$, then $p(z)$ has all roots on the unit circle.

Lemma 5.2.5. *For any $b \in \mathbb{N}$, the polynomial $f_b(t) = 2\sum_{i=1}^{2b} t^i + t^b$ is irreducible over $\mathbb{Q}[t]$ and not a norm in $\mathbb{C}[t]$.*

Proof. First, observe that $f_b(t)$ satisfies the hypotheses of Theorem 5.2.4, since we have $l = 2$, $a_k = 0$ for $k = 0, \ldots, b-1$, and $a_b = 1$. So for any $b \in \mathbb{N}$, the polynomial $f_b(t)$ has all of its roots on the unit circle.

Since $f_b(t)$ is symmetric, there is $l_b(t) \in \mathbb{R}[t]$ such that $f_b(t) = l_b\left(t + \frac{1}{t}\right)$. However, since $f_b(t)$ has only unit norm roots, any factor of $f_b(t)$ over $\mathbb{Q}[t] \subset \mathbb{R}[t]$ must be symmetric and so of the form $g\left(t + \frac{1}{t}\right)$ for some $g(t)$ dividing $l_b(t)$. In particular, in order to show that $f_b(t)$ is irreducible in $\mathbb{Q}[t]$ it suffices to show that $l_b(t)$ is irreducible in $\mathbb{Q}[t]$. Now note that $l_b(t) = \sum_{j=0}^{k=b} a_j t^j$ must have $a_b = 2$, a_j even for $0 < j < b$, and a_0 odd. Therefore, by Eisenstein's criterion with $p = 2$ and Gauss's Lemma, the integral polynomial $t^b l_b(t^{-1})$ is irreducible over $\mathbb{Q}[t]$ and so $l_b(t)$ and $f_b(t)$ are irreducible as well. Since $f_b(t)$ is irreducible, its roots are distinct. In particular, $f_b(t)$ has at least one complex root of unit norm with odd multiplicity.

Now let $t^k g(t)\overline{g(t^{-1})}$ be any norm in $\mathbb{C}[t]$. Note that if α is a nonzero root of $g(t)$ then $\overline{\frac{1}{\alpha}}$ is a root of $\overline{g(t^{-1})}$. In particular, if α is a unit norm root of $g(t)$, then $\alpha = \overline{\frac{1}{\alpha}}$ is a root of $\overline{g(t^{-1})}$ of the same multiplicity. That is, any norm in $\mathbb{C}[t]$ must have all unit-norm roots occurring with even multiplicity and so $f_b(t)$ is not a norm. $\square$

Almost identical arguments show that $f_b^*(t)$ is irreducible, relatively prime to $g_n^*(t)$, and not a norm, and hence that the reduced twisted Alexander polynomial for $K_{m,n}^-$ constructed via $\tilde{\rho}^*$ and χ^* is not a norm in $\mathbb{C}[t^{\pm 1}]$.

We are now ready to prove Theorem 5.1.2.

Proof of Theorem 5.1.2. First, note that since $P(2n, m, -2n-1, -m)$ and $P(2n, -m, -2n-1, m)$ are the same as unoriented knots, we can assume without loss of generality that $m > 0$. By Lemma 5.2.1 we have that the module $H_1(\Sigma_p(K), \mathbb{F}_2)$ is irreducible. Therefore, as observed by [20], any metabolizer $M \leq H_1(\Sigma_p(K))$ must have trivial image in $H_1(\Sigma_p(K), \mathbb{F}_2)$. It follows that any map $H_1(X_p(K)) \to \mathbb{F}_2$ that factors through $H_1(\Sigma_p(K))$ vanishes on M. Therefore, to obstruct K's sliceness it suffices to show that there is some such map such that the corresponding reduced twisted Alexander polynomial is not a norm in $\mathbb{C}[t^{\pm 1}]$ and hence not in a norm in any $\mathbb{Q}(\xi_{2^k})[t^{\pm 1}]$. In the following, we construct this map, compute the corresponding reduced twisted Alexander polynomial explicitly (Lemma 5.2.2), and show that this polynomial is not a norm in $\mathbb{C}[t^{\pm 1}]$ (Lemma 5.2.3), except when $n = 3$ and $p = 5$. $\square$

5.3 Additional examples

We now consider two examples of 4-strand pretzels which do not fit the hypotheses of Theorem 5.1.2. In particular, these knots are of the form $K =$

$P(2a, p, -(2a \pm 1), -p)$ and have the property that $H_1(\Sigma_p(K), \mathbb{F}_2)$ contains nontrivial proper submodules. We will see that it is still sometimes possible (with a little more work) to obtain obstructions to topological sliceness in these cases.

Example 5.3.1. Let $K_1 = P(6, 3, -7, -3)$. Via the arguments of Lemma 5.2.1 one can show that $H = H_1(\Sigma_3(K_1), \mathbb{F}_2)$ is isomorphic to $V_1 \oplus V_1$ as an $\mathbb{F}_2[x]$-module, where $V_1 = \mathbb{F}_2[x]/\langle x^2 + x + 1\rangle$. In particular, H has 5 distinct potential metabolizers (i.e. submodules of square root order), each of which is generated as an $\mathbb{F}_2[x]$-module by a single element.[6] Observe that any map $\pi_1(X_3(K_1)) \to H \to \mathbb{F}_2$ that vanishes on a metabolizer M certainly factors through a map $\pi_1(X_3(K_1)) \to H/M \to \mathbb{F}_2$. It is also easy to check that for each potential metabolizer M, the module H/M is isomorphic to V_1. Therefore, it certainly suffices to show that for any nontrivial choices of $\rho\colon \pi_1(X_3(K_1)) \to H_1(\Sigma_3(K_1)) \to V_1$ and $\chi\colon V_1 \to \mathbb{F}_2$, the corresponding reduced twisted Alexander polynomial $\widetilde{\Delta}_{X_3(K_1), \epsilon_3 \otimes \phi_{\chi \circ \rho}}(t)$ does not factor as a norm over $\mathbb{Q}(\xi_{2^N})[t^{\pm 1}]$ for any $N \in \mathbb{N}$.

As before, we use the results of Herald, Kirk, and Livingston in [20] to establish a correspondence between $\rho\colon \pi_1(X_3(K_1)) \to H_1(\Sigma_3(K_1)) \to V_1$ and $\tilde{\rho}\colon \pi_1(X(K_1)) \to \mathbb{Z} \ltimes V_1$. We consider all such nontrivial maps and all nontrivial choices of $\chi\colon V_1 \to \mathbb{F}_2$ and compute the twisted Alexander polynomial associated to each of these choices. We obtain only four distinct reduced twisted Alexander polynomials, listed as products of irreducible polynomials in $\mathbb{Q}[t^{\pm 1}]$:

- $f_1(t) = (2t^2 - t + 2)(2t^2 + 3t + 2)$

[6] Five such $\mathbb{F}_2[x]$-generators are $(1, 0), (0, 1), (1, 1), (1, x)$, or $(1, x^2)$.

- $f_2(t) = (4t^4 - 8t^3 + 7t^2 - 8t + 4)$
- $f_3(t) = (8t^6 + 7t^4 - 14t^3 + 7t^2 + 8)$
- $f_4(t) = (4t^2 - 11t + 4)$.

In the first three cases, we can use arguments similar to those of Lemma 5.2.5 to show that $f_i(t)$ is not even a norm over $\mathbb{C}[t^{\pm 1}]$. However, $f_4(t)$ has roots $\alpha_\pm = \frac{11 \pm \sqrt{57}}{8}$ and hence is a norm over $\mathbb{C}[t^{\pm 1}]$, so a little more work is required. In particular, note that $f_4(t)$ factors as a norm over an extension E of $\mathbb{Q}$ if and only if $\sqrt{57} \in E$ and so we must show that $\sqrt{57}$ is never in $\mathbb{Q}(\xi_{2^N})$. This follows immediately from the next claim.

Claim: There are only three quadratic extensions of the rationals which are contained in any $\mathbb{Q}(\xi_{2^N})$: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{2}i)$.

First, note that for $N \geq 3$ the listed fields certainly are contained in $\mathbb{Q}(\xi_{2^N})$. Now, let $N \geq 3$ be given in order to show that these are the only ones. By the Fundamental Theorem of Galois theory, quadratic subextensions of $\mathbb{Q}(\xi_{2^N})$ are in bijective correspondence with index 2 subgroups of $Gal(\mathbb{Q}(\xi_{2^N}), \mathbb{Q}) \cong \left(\mathbb{Z}/2^N\mathbb{Z}\right)^\times$. It is a standard exercise in group theory to show that $\left(\mathbb{Z}/2^N\mathbb{Z}\right)^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{N-2}}$. Finally, one can use Goursat's Lemma (which describes the subgroups of a direct product) to show that $\mathbb{Z}_2 \times \mathbb{Z}_{2^{N-2}}$ has exactly 3 subgroups of index 2: $1 \times \mathbb{Z}_{2^{N-2}}$, $\mathbb{Z}_2 \times \mathbb{Z}_{2^{N-3}}$, and $\{(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2^{N-2}} : a \equiv b \mod 2\}$. Therefore, $\mathbb{Q}(\xi_{2^N})$ has exactly 3 subextensions which are degree 2 over $\mathbb{Q}$ and so our list is complete.

Example 5.3.2. We now consider $K_2 = P(8, 7, -9, -7)$, where (since 2 is not a primitive root mod 7) we have that $H_1(\Sigma_7(K_2), \mathbb{F}_2)$ is the cyclic but not irreducible $\mathbb{F}_2[x]$-module $V_2 = \mathbb{F}_2[x]/\langle x^6+x^5+x^4+x^3+x^2+x+1\rangle \cong \mathbb{F}_2[x]/\langle x^3+$

$x^2+1\rangle \oplus \mathbb{F}_2[x]/\langle x^3+x+1\rangle$. Instead of constructing maps vanishing on each of the 7 potential metabolizers, computing the associated polynomials, and obstructing their factorization, we consider the twisted Alexander polynomials associated to the double cover instead.[7]

Arguments as in Lemma 5.2.1 show that $H_1(\Sigma_2(K_2), \mathbb{F}_7) \cong \mathbb{F}_7[x]/\langle x+1\rangle$ is an irreducible $\mathbb{F}_7[\mathbb{Z}_2]$ module. There is a nontrivial map $\pi_1(X_2(K_2)) \to H_1(\Sigma_2(K_2), \mathbb{F}_7) \to \mathbb{Z}_7$, unique up to rescaling, and the single corresponding reduced twisted Alexander polynomial is the product of $(t-1)^2$ and a degree 22 polynomial $g(t) \in \mathbb{Z}(\xi_7)[t^{\pm 1}]$. We use the following extension of Gauss' Lemma due to Herald, Kirk, and Livingston to show that $g(t)$ is not a norm over $\mathbb{Q}(\xi_7)$.

Lemma 5.3.3 ([20])**.** *Let k and r be primes such that $r = nk+1$ for some $n \in \mathbb{N}$. Let $b \in \mathbb{Z}_r$ be a nontrivial k^{th} root of 1, and let $\phi\colon \mathbb{Z}[\xi_k] \to \mathbb{Z}_r$ be the ring homomorphism sending 1 to 1 and ξ_k to b. Let $p(t) \in \mathbb{Z}[\xi_k](t)$ be a degree $2m$ polynomial, such that $\phi(p(t))$ also has degree $2m$. If $p(t)$ is a norm in $\mathbb{Q}(\xi_k)(t)$, then $\phi(p(t))$ factors as the product of two degree m polynomials in $\mathbb{Z}_r[t]$.*

In particular, note that 16 is a 7^{th} root of 1 modulo $29 = 4 \cdot 7 + 1$. Under the map $\xi_7 \to 16$, the polynomial $g(t)$ maps to a degree 22 polynomial, whose irreducible factorization over $\mathbb{Z}_{29}$ is given by 10 linear factors and a single degree 12 irreducible polynomial, $\hat{g}(t) = 1 + 13 + 19t^2 + 13t^3 + 9t^4 +$

[7] This may at first seem surprising, since K_2 is the mutant of a ribbon knot and so we cannot expect any sliceness obstructions from its double branched cover. However, this serves to emphasize the fact that, despite the requirement that we choose a character factoring through the homology of a branched cover, the twisted Alexander polynomials of a knot are really invariants of *unbranched* covers (and of course a knot and its mutant will not generally have the same unbranched double cover).

$7t^5 + 22t^6 + 7t^7 + 9t^8 + 13t^9 + 19t^{10} + 13t^{11} + t^{12}$. So the image of $g(t)$ does not factor as the product of two degree 11 polynomials over $\mathbb{Z}_{29}$ and hence by Lemma 5.3.3 is not a norm over $\mathbb{Q}(\xi_7)$.

5.4 Matrix computations

The remaining results are primarily consequences of matrix manipulation.

Lemma 5.4.1. *Let $k = \frac{p+1}{2} + jp$ and $n \in \mathbb{N}$. Let $A_n, C_k, D_{k,n}$, and E_k be defined as before. Then the following hold:*

1. $\det(E_k) = \det(C_k)$
2. $F_{k,n} := E_k^{-1}(D_{k,n} - y(xy)^k A_n)$ *is independent of k.*

Proof. First, observe that $y = axa$, where a is a diagonal matrix with entries $a_{i,i} = \begin{cases} -1 & \text{if } i = 1,2 \\ 1 & \text{else} \end{cases}$. Therefore, $(xy)^{\frac{p-1}{2}}x = (xaxa)^{\frac{p-1}{2}}x = (xa)^p a$ and $y(xy)^{\frac{p-1}{2}} = axa(xaxa)^{\frac{p-1}{2}} = a(xa)^p$. Since $(xa)^p$ can be easily computed to be the diagonal matrix tI_p, we have that $a(xa)^p = (xa)^p a$ and hence $(xy)^{\frac{p-1}{2}}x = y(xy)^{\frac{p-1}{2}}$. It also follows that $(xy)^{ip} = (xa)^{2ip} = t^{2i}I_p = (yx)^{ip}$ for any $i \in \mathbb{N}$. Therefore, recalling that $k = jp + \frac{p-1}{2}$, we have the following equivalent expressions for $(xy)^k x$.

$$(xy)^{jp}(xy)^{\frac{p-1}{2}}x = t^{2j}(xy)^{\frac{p-1}{2}}x = y(xy)^{\frac{p-1}{2}}t^{2j} = y(xy)^{\frac{p-1}{2}}(yx)^{jp} = y(xy)^k.$$

Now observe that

$$\begin{aligned} E_k(1 - xy) &= 1 - xy + (x-1)y(1 - (xy)^k) = 1 + y(xy)^k - y - (xy)^{k+1} \\ &= 1 + y(xy)^k - (1 + (xy)^k x)y = 1 + y(xy)^k - (1 + y(xy)^k)y \\ &= (1 + y(xy)^k)(1 - y). \end{aligned}$$

Similarly,

$$\begin{aligned} C_k(1-yx) &= 1 + x(yx)^k - x - (yx)^{k+1} = 1 + x(yx)^k - (1 + y(xy)^k)x \\ &= 1 + y(xy)^k - (1 + y(xy)^k)x = (1 + y(xy)^k)(1-x). \end{aligned}$$

The matrices x and y are invertible and so $\det(1-xy) = \det(1-yx)$. We can also explicitly check that $\det(1-xy) \neq 0$ and $\det(1-x) = \det(1-y)$ and conclude that $\det(C_k) = \det(E_k)$.

It also follows that $E_k^{-1}C_k$ and $E_k^{-1}(1 + y(xy)^k)$ are independent of k, since by the above

$$\begin{aligned} E_k^{-1}C_k &= (1-xy)(E_k(1-xy))^{-1}C_k(1-yx)(1-yx)^{-1} \\ &= (1-xy)(1-y)^{-1}(1+y(xy)^k)^{-1}(1+y(xy)^k)(1-x)(1-yx)^{-1} \\ &= (1-xy)(1-y)^{-1}(1-x)(1-yx)^{-1}. \end{aligned}$$

and

$$\begin{aligned} E_k^{-1}(1+y(xy)^k) &= (1-xy)(1-y)^{-1}(1+y(xy)^k)^{-1}(1+y(xy)^k) \\ &= (1-xy)(1-y)^{-1}. \end{aligned}$$

Finally, observe that $(*) = D_{k,n} - y(xy)^k A_n$ has the following descriptions.

$$\begin{aligned} (*) &= \sum_{i=0}^{2n+1}(-y)^i + (y-1)x\sum_{i=0}^{k-1}(yx)^i(-y)^{2n+1} + y(xy)^k\sum_{i=0}^{2n}(-y)^i \\ &= (1+y(xy)^k)\sum_{i=0}^{2n}(-y)^i + \left(1 + (y-1)x\sum_{i=0}^{k-1}(yx)^i\right)(-y)^{2n+1} \\ &= -(1+y(xy)^k)A_n - C_k y^{2n+1}. \end{aligned}$$

Hence $F_{k,n} = E_k^{-1}(D_{k,n} - y(xy)^k A_n) = -E_k^{-1}(1+y(xy)^k)A_n - E_k^{-1}C_k y^{2n+1}$ is independent of k as well. $\square$

Lemma 5.4.2. *Let p be prime and $n \in \mathbb{N}$ such that $2n = bp + a$ for $0 < a < p-1$ and $b \geq 1$. Then $\det(A_n + B_nF_n) \doteq f_b(t)\det(G_n)(1+t)^{-1}$, where $f_b(t)$ is as in Lemma 5.2.2 and*

$$G_n := \begin{bmatrix} (p-a-2)\beta_b(t) & -1 & -1 \\ \Psi_b(t) & 2(-1)^b & -2t^{b+1} \\ (a-2)\beta_{b+1}(t) & 1 & -t \end{bmatrix}, \text{ where}$$

$$\beta_b(t) = 2\sum_{i=1}^{b}(-t)^i \text{ and } \Psi_b(t) = (-1)^b t\left(2\sum_{i=0}^{2b}(-t)^i + (-t)^b\right)$$

Proof. First, observe that when $p = 3$ or $p = 5$ the matrix $A_n + B_nF_n$ is of small size and one can explicitly compute the form above, with minimal simplification required. So suppose $p \geq 7$. Observe that $A_n(1+y) = -(1+y^{2n+1})$, so we will begin by considering the matrix

$$\begin{aligned} -(A_n + B_nF_n)(1+y) &= -A_n + B_nE_0^{-1}(C_0y^{2n+1} + (1+y(xy)^{\frac{p-1}{2}})A_n)(1+y) \\ &= 1 + y^{2n+1} + B_n\left(E_0^{-1}C_0(1+y)y^{2n+1} - E_0^{-1}(1+y(xy)^{k_0})(1+y^{2n+1})\right) \end{aligned}$$

We can compute $E_0^{-1}C_0$ and $E_0^{-1}(1+y(xy)^{k_0})$ using the expressions from Lemma 5.4.1. Also note that $(1+x)B_n = 1 - x^{2n}$ is also easily computable, leading us to an easy verification for the form of B_n.

Combining these expressions, when $1 < a < p-2$ we get the following form for $(-1)(A_n + B_nF_n)(1+y)$, where similar expressions hold for $a = 1$ and $a = p-2$.

$$
\left[\begin{array}{cc|ccccc|cc|ccc}
-\gamma_b(t) & -\gamma_b(t) & \alpha_b(t) & \cdots & & \alpha_b(t) & \alpha_b(t)-t^b & \epsilon_b(t) & \epsilon_b(t)-t^b & \alpha_{b-1}(t) & \cdots & \alpha_{b-1}(t) \\
\gamma_b(t) & \gamma_b(t) & -\alpha_b(t) & \cdots & & -\alpha_b(t) & -\alpha_b(t) & -\epsilon_b(t) & -\epsilon_b(t)+2t^b & -\alpha_{b-1}(t) & \cdots & -\alpha_{b-1}(t) \\
\hline
\beta_b(t) & \beta_b(t) & & & & & & t^b\beta_b(t) & t^b\beta_b(t)-t^b & -t^b & \cdots & 0 \\
-\beta_b(t) & -\beta_b(t) & & & & & & -t^b\beta_b(t) & -t^b\beta_b(t) & -t^b & \cdots & 0 \\
\vdots & \vdots & \multicolumn{5}{c|}{0_{p-a-2,a-1}} & \vdots & \vdots & \vdots & \ddots & 0 \\
\beta_b(t) & \beta_b(t) & & & & & & t^b\beta_b(t) & t^b\beta_b(t) & 0 & \cdots & -t^b \\
-\beta_b(t)+t^{b+1} & -\beta_b(t) & & & & & & -t^b\beta_b(t) & -t^b\beta_b(t) & 0 & \cdots & -t^b \\
\hline
\eta_b(t) & \eta_b(t)+2t^{b+1} & -\phi_b(t) & -\phi_b(t) & & \cdots & -\phi_b(t) & -4t^{2b} & -4t^{2b} & -\theta_b(t) & \cdots & -\theta_b(t) \\
-\eta_b(t)-t^{b+1} & -\eta_b(t)-2t^{b+1} & \phi_b(t)-t^{b+1} & \phi_b(t) & & \cdots & \phi_b(t) & 4t^{2b} & 4t^{2b} & \theta_b(t) & \cdots & \theta_b(t) \\
\hline
\beta_{b+1}(t) & \beta_{b+1}(t) & -t^{b+1} & -t^{b+1} & 0 & \cdots & 0 & t^b\beta_{b+1}(t) & t^b\beta_{b+1}(t) & & & \\
-\beta_{b+1}(t) & -\beta_{b+1}(t) & 0 & t^{b+1} & -t^{b+1} & \cdots & 0 & -t^b\beta_{b+1}(t) & -t^b\beta_{b+1}(t) & \multicolumn{3}{c}{0_{a-2,p-a-3}} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & & \\
\beta_{b+1}(t) & \beta_{b+1}(t) & 0 & 0 & \cdots & -t^{b+1} & -t^{b+1} & t^b\beta_{b+1}(t) & t^b\beta_{b+1}(t) & & &
\end{array}\right]
$$

The matrix $-(A_n + B_nF_n)(1+y)$, written as a block matrix with dimensions $(2+(p-a-2)+2+(a-2), 2+(a-1)+2+(p-a-3))$.

$$\text{where } \alpha_b(t) := 2\sum_{i=0}^{b} t^i,\ \beta_b(t) := 2\sum_{i=1}^{b}(-t)^i,\ \gamma_b(t) := 4\sum_{i=0}^{\lceil \frac{b}{2}\rceil} t^{2i+1},\ \eta_b(t) := 2\sum_{i=b+2}^{2b+1} t^i + t^{b+1} + (-1)^{b+1}2\sum_{i=1}^{b}(-t)^i,$$

$$\epsilon_b(t) := (-1)^b 2\sum_{i=b+1}^{2b}(-t)^i + 3t^b + 2\sum_{i=0}^{b-1} t^i,\ \theta_b(t) := t^{b+1}\alpha_{b-1}(t),\ \text{ and } \phi_b(t) := \theta_b(t) + 2t^{2b+1} = t^{b+1}\alpha_b(t)$$

Examination of the alternating signs above indicates that the form above applies only when a is odd. The form when a is even is exactly analogous and omitted.

Some easy row and column moves[8] let us rewrite this matrix as follows:

$$\left[\begin{array}{c|c|c|c}
\begin{array}{cc} -\gamma_b(t) & 0 \\ 0 & 0\end{array} & \begin{array}{cccc}\alpha_b(t) & \cdots & \alpha_b(t) & \alpha_b(t)-t^b \\ 0 & \cdots & 0 & -t^b\end{array} & \begin{array}{cc} f_b(t) & -t^b \\ 0 & t^b\end{array} & \begin{array}{cccc}\alpha_{b-1}(t) & \alpha_{b-1}(t) & \cdots & \alpha_{b-1}(t)\\ 0&0&\cdots&0\end{array} \\ \hline
\begin{array}{cc}\beta_b(t)&0\\-\beta_b(t)&0\\ \vdots&\vdots\\ \beta_b(t)&0\\-\beta_b(t)&-t^{b+1}\end{array} & 0_{p-a-2,a-1} & \begin{array}{cc}0&-t^b\\0&0\\ \vdots&\vdots\\0&0\\0&0\end{array} & \begin{array}{cccc}-t^b&0&\cdots&0\\-t^b&-t^b&\cdots&0\\0&\ddots&\ddots&0\\0&\cdots&-t^b&-t^b\\0&\cdots&0&-t^b\end{array}\\ \hline
\begin{array}{cc}\Psi_b(t)&2t^{b+1}\\0&t^{b+1}\end{array} & \begin{array}{ccccc}0&0&\cdots&0&-t^{2b+1}\\-t^{b+1}&0&\cdots&0&0\end{array} & \begin{array}{cc}0&-t^{2b+1}\\0&0\end{array} & \begin{array}{cccc}0&0&\cdots&0\\0&0&\cdots&0\end{array}\\ \hline
\begin{array}{cc}\beta_{b+1}(t)&0\\-\beta_{b+1}(t)&0\\ \vdots&\vdots\\ \beta_{b+1}(t)&0\end{array} & \begin{array}{ccccc}-t^{b+1}&-t^{b+1}&0&\cdots&0\\0&t^{b+1}&-t^{b+1}&\cdots&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\cdots&-t^{b+1}&-t^{b+1}\end{array} & \begin{array}{cc}0&0\\0&0\\ \vdots&\vdots\\0&0\end{array} & 0_{a-2,p-a-3}
\end{array}\right]$$

where $\alpha_b(t) := 2\sum_{i=0}^{b} t^i$, $\beta_b(t) := 2\sum_{i=1}^{b}(-t)^i$, $\gamma_b(t) := 4\sum_{i=0}^{\lceil \frac{b}{2}\rceil} t^{2i+1}$, $\eta_b(t) := 2\left(\sum_{i=b+2}^{2b+1} t^i(-1)^{b+1}\sum_{i=1}^{b}(-t)^i\right) + t^{b+1}$,

$$\epsilon_b(t) := (-1)^b 2\sum_{i=b+1}^{2b}(-t)^i + 3t^b + 2\sum_{i=0}^{b-1} t^i,\ \Psi_b(t) := \eta_b(t) + 2t^{b+1} - t^{b+1}\gamma_b(t)$$

Note that $f_b(t) = 2\sum_{i=0}^{2b} t^i + t^b$ is obtained in the above matrix as $f_b(t) = \epsilon_b(t) + t^b\gamma_b(t)$.

[8]To be specific, perform the following operations, in this order: add r_1 to r_2, add r_{p-a+1} to r_{p-a+2}, add $-c_1$ to c_2, add $-c_{a+2}$ to c_{a+3}, add $-t^b c_1$ to c_{a+2}, add c_2 to c_1, add $-t^b c_2$ to c_{a+2}, and add $t^{b+1} r_1$ to r_{p-a-1}.

Therefore, $\det(A_n + B_nF_n) \doteq f_b(t)\det(M_n)(1+t)^{-1}$, where M_n is obtained from the previous matrix by the deletion of rows 1, 2 and columns $p-a+1, p-a+2$ and moving a column.

$$
M_n = \left[\begin{array}{ccc|c|c}
\beta_b(t) & 0 & -t^b & & \\
-\beta_b(t) & 0 & 0 & & \\
\vdots & \vdots & \vdots & 0_{p-a-2,a-2} & \mathrm{E}^b_{p-a-2} \\
\beta_b(t) & 0 & 0 & & \\
-\beta_b(t) & -t^{b+1} & 0 & & \\
\hline
\Psi_b(t) & 2t^{b+1} & -2t^{2b+1} & 0 \ \cdots \ 0 & 0 \ \cdots \ 0 \\
\hline
0 & t^{b+1} & 0 & & \\
\beta_{b+1}(t) & 0 & 0 & & \\
\vdots & \vdots & \vdots & \mathrm{E}^{b+1}_{a-1} & 0_{a,p-a-3} \\
-\beta_{b+1}(t) & 0 & 0 & & \\
\beta_{b+1}(t) & 0 & -t^{b+1} & &
\end{array}\right]_{p-2,p-2} \tag{5.3}
$$

where

$$
E^b_k := \begin{bmatrix}
-t^b & 0 & 0 & \cdots & 0 \\
-t^b & -t^b & 0 & \cdots & 0 \\
0 & -t^b & -t^b & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -t^b & -t^b \\
0 & \cdots & 0 & 0 & -t^b
\end{bmatrix}_{k,k-1}.
$$

Note that each of the columns $c_4, \ldots, c_{p-2}$ of M_n contain exactly two nonzero entries. We can apply simple row moves to show that $\det(M_n) \doteq \det(G_n)$, where

$$
G_n := \begin{bmatrix}
(p-a-2)\beta_b(t) & -1 & -1 \\
\Psi_b(t) & 2(-1)^b & -2t^{b+1} \\
(a-2)\beta_{b+1}(t) & 1 & -t
\end{bmatrix}
$$

Finally, note that

$$\begin{aligned}\Psi_b(t) &= \eta_b(t) + 2t^{b+1} - t^{b+1}\gamma_b(t) \\ &= 2\sum_{i=b+2}^{2b+1} t^i + 3t^{b+1} + (-1)^{b+1}2\sum_{i=1}^{b}(-t)^i - 4t^{b+1}\sum_{i=1}^{\lceil \frac{b}{2}\rceil} t^{2i+1} \\ &= t\left((-1)^b \sum_{i=b+1}^{2b} 2(-t)^i + 3t^b + (-1)^b\sum_{i=0}^{b-1}2(-t)^i\right) \\ &= (-1)^b t\left(2\sum_{i=0}^{2b}(-t)^i + (-t)^b\right), \text{ as desired.}\end{aligned}$$

□

5.4.1 Sample computations of $f_b(t)$ and $g_n(t)$.

Finally, we give some computations of $f_b(t)$ and $g_n(t)$, normalized to have positive leading coefficient. Observe that when $(n,p) = (3,5)$ we have that $f_b(t) = g_n(t)$ and so the associated twisted Alexander polynomial is $f_b(t)g_n(t)h_k(t)^2(t-1)^{-2}$, and therefore is certainly a norm.

n	(b,a)	$f_b(t)$	$g_n(t)$
6	$(1,1)$	$2t^2+3t+2$	$2t^2+27t+2$
7	$(1,3)$	$2t^2+3t+2$	$6t^2-35t+6$
8	$(1,5)$	$2t^2+3t+2$	$14t^2-43t+14$
9	$(1,7)$	$2t^2+3t+2$	$22t^2-51t+22$
10	$(1,9)$	$2t^2+3t+2$	$30t^2-59t+30$
12	$(2,2)$	$2t^4+2t^3+3t^2+2t+2$	$2t^4-30t^3+59t^2-30t+2$
13	$(2,4)$	$2t^4+2t^3+3t^2+2t+2$	$10t^4-38t^3+67t^2-38t+10$
14	$(2,6)$	$2t^4+2t^3+3t^2+2t+2$	$18t^4-46t^3+75t^2-46t+18$

Table 5.1: Some computations of $f_b(t)$ and $g_n(t)$, with $p = 11$ and $2n = bp+a$.

n	(b,a)	$g_n(t)$
3	$(1,1)$	$2t^2+3t+2$
4	$(1,3)$	$6t^2-11t+6$
6	$(2,2)$	$2t^4-6t^3+11t^2-6t+2$
8	$(3,1)$	$2t^6+2t^5-6t^4+11t^3-6t^2+2t+2$
9	$(3,3)$	$6t^6-10t^5+14t^4-19t^3+14t^2-10t+6$

Table 5.2: More computations of $g_n(t)$, with $p=5$ and $2n=bp+a$.

Chapter 6

Reversal and concordance

We prove that given any patterns P and Q of opposite winding number, for any $n \geq 0$ there exists a knot K such that the minimal genus of a cobordism between $P(K)$ and $Q(K)$ is at least n. This answers a question posed by Cochran-Harvey [CH17] and generalizes a result of Kim-Livingston [KL05].

6.1 Introduction

While most of the investigations of $\mathcal{C}$, the collection of knots modulo concordance, have focused on its group structure, it is also natural to consider it as a metric space with metric $d(K,J) := g_4(K\# -J)$. Cochran and Harvey [7] considered this geometric structure, focusing on the metric properties of maps induced by patterns in solid tori. Following their work, we consider the distance between two patterns, defined as

$$d(P,Q) = \sup_{K\in\mathcal{C}} d(P(K),Q(K)) \in \{0,1,2,\ldots,\infty\}.$$

It is natural to ask when two patterns are a finite distance from each other. Cochran and Harvey use Tristram-Levine signatures to give an almost complete characterization of this in terms of winding number. (For a discussion of pattern orientations, including a definition of winding number, we refer the reader to Section 6.2.) All results stated here hold in both the smooth and the topological categories, since the constructions are smooth and the obstructions are topological.

Theorem 6.1.1 (Cochran-Harvey [7]). *Let P and Q be patterns of winding number m and n, respectively. If $n = m$, then $d(P, Q)$ is finite and if $|n| \neq |m|$, then $d(P, Q)$ is infinite.*

We are therefore led to consider whether the distance between a winding number m pattern and a winding number $-m$ pattern can ever be finite. Cochran-Harvey's arguments do not apply in this case: Tristram-Levine signatures are insensitive to the orientation of a knot, and for every winding number m pattern P there is a winding number $-m$ pattern P^r such that $P(K)$ and $P^r(K)$ are always equal as unoriented knots. Nevertheless, the case of $m = 1$ was resolved by Kim and Livingston [23] by using Casson-Gordon invariants, in a result that seems undeservedly forgotten. Note that the core of the torus, oriented one way, gives a winding number 1 satellite map $K \mapsto K$ and, oriented the other way, gives a winding number -1 satellite map $K \mapsto K^r$.

Theorem 6.1.2 (Kim-Livingston [23]). *For any $g \geq 0$ there exists a knot K such that $g_4(K\# - K^r) > g$. That is, the identity (winding number +1) and reversal (winding number -1) operators are infinite distance from each other.*

It seems to have been assumed that the extension of Theorem 6.1.2 to the case of general $m > 0$ would require substantial advances in the computation of Casson-Gordon invariants (see e.g. Remark 6.15 of [7]). It is therefore perhaps somewhat surprising that we prove the following result while computationally only using Litherland's work of [34]; on the other hand, the potential relevance of formulae for Casson-Gordon invariants of satellite knots to the problem is clear.

Theorem 6.1.3. *Let $m > 0$ and P and Q be patterns of winding number m and $-m$, respectively. Then $d(P, Q)$ is infinite.*

Theorems 6.1.1 and 6.1.3 combine to give the following.

Corollary 6.1.4. *Let P and Q be patterns of winding number m and n, respectively. Then the distance between P and Q is finite if and only if $m = n$.*

6.2 Background

As discussed in Section 2.2, to an oriented knot K and and a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ on the first homology of the nth cyclic branched cover of K, Casson-Gordon associate a rational number $\sigma_1\,\tau(K, \chi)$, which is roughly the twisted signature of some associated 4-manifold [4]. We have the following key proposition relating the Casson-Gordon signatures of a knot to those of its mirror image (i.e. the concordance inverse of its reverse), which follows immediately from the basic definitions.

Proposition 6.2.1. *Let K be a knot, $-K^r$ denote its mirror image, and $n \in \mathbb{N}$. Then there is a canonical isomorphism of groups $\alpha\colon H_1(\Sigma_n(K)) \to H_1(\Sigma_n(-K^r))$ such that*

1. *Letting t_K and t_{-K^r} denote the actions induced by the natural covering transformations on $H_1(\Sigma_n(K))$ and $H_1(\Sigma_n(-K^r))$, respectively, we have $t_{-K^r} \cdot \alpha(x) = \alpha(t_K^{-1} \cdot x)$ for all $x \in H_1(\Sigma_n(K))$.*

2. *Given $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_m$ we have $\sigma_1\,\tau(-K^r, \chi \circ \alpha^{-1}) = -\,\sigma_1\,\tau(K, \chi)$.*

Notice that if we replace $-K^r$ with $-K$, Part (1) of Proposition 6.2.1 would be replaced with $t_{-K} \cdot \alpha(x) = \alpha(t_K \cdot x)$; since Casson-Gordon signatures are additive with respect to connected sums of knots, we would not be able to obtain any potential slice genus obstruction for $K\# -K$. This is reassuring, since $K\# -K$ is of course always slice. It also suggests to us that in order to obtain lower bounds for $g_4(K\# -K^r)$, we must pay particular attention to the action on the first homology induced by the covering transformation. We will use Gilmer's slice genus bound, in a slightly different form than originally stated. We use $\sigma_K(\omega)$ to denote the Tristram-Levine signature of a knot K at $\omega \in S^1$ and for $n \in \mathbb{N}$ let $\omega_n := e^{2\pi i/n}$.

Theorem 6.2.2 (Gilmer [17]). *Let K be a knot and suppose that $g_4(K) \leq g$. Then for any prime power n there is a decomposition of $H_1(\Sigma_n(K)) \cong A_1 \oplus A_2$ so that the following properties hold:*

1. *A_1 has a rank $2(n-1)g$ presentation with signature equal to $\sum_{i=1}^{n} \sigma_K(\omega_n^i)$.*

2. *A_2 has a subgroup B such that $|B|^2 = |A_2|$ and for any prime power order character $\chi : H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ which vanishes on $A_1 \oplus B$, we have*
$$|\,\sigma_1\, \tau(K,\chi) + \sum_{i=1}^{n} \sigma_K(\omega_n^i)| \leq 2ng.$$
Also, $A_1 \oplus B$ and B are both covering transformation invariant.

Proof. This follows from Gilmer's proof. Letting W_n denote the n-fold cyclic branched cover of the 4-ball over the hypothesized genus g surface with boundary K and abbreviating $\Sigma_n = \Sigma_n(K)$, we obtain A_1 and B from the following exact sequence:

$$0 \to H_2(W_n) \to H_2(W_n, \Sigma_n) \xrightarrow{\partial} H_1(\Sigma_n) \to H_1(W_n) \to H_1(W_n, \Sigma_n) \to 0.$$

In particular, $A_1 \oplus B = \mathrm{im}(\partial)$ and $B = \mathrm{im}\left(\partial|_{TH_2(W_n,\Sigma_n)}\right)$ are covering transformation invariant. $\square$

Note that Gilmer's original proof did not include any consideration of covering transformation invariance, due perhaps to the fact that his work explicitly dealt with the case $n = 2$, when t acts by multiplication by -1 and all subgroups are covering transformation invariant. Kim and Livingston's [23] proof that there exist knots K for which $g_4(K\#-K^r)$ is arbitrarily large relies on this more general result in the case $n = 3$.

Our examples are constructed via various satellite operations. Given the importance of orientation in our context, we rather pedantically establish some orientation conventions pertaining to patterns in solid tori. Choose fixed orientations on S^1 and D^2. These induce orientations on $V := S^1 \times D^2$ and $\lambda_V := S^1 \times \{x_0\}$, where x_0 is a (positively oriented) point in ∂D^2, as well as on $\mu_V := \{y_0\} \times \partial D^2$. These orientations for V, λ_V, and μ_V will remain fixed throughout. Given a pattern $P : S^1 \to V$, the class of $P(S^1)$ in $H_1(V)$ is equal to $n[\lambda_V]$ for some $n \in \mathbb{Z}$. We call n the *(algebraic) winding number* of P. To an oriented knot K in S^3 we associate the positively oriented meridian μ_K and 0-framed longitude λ_K in the standard way. Finally, note that as usual we mildly abuse notation by, for example, referring to both the map P and its image $P(S^1)$ as P.

Definition 6.2.3. Given a knot K in S^3 and a pattern $P : S^1 \to V$, define the *satellite knot* $P(K)$ as follows: Let $i_K\colon V \to \overline{\nu(K)} \subset S^3$ be a homeomorphism with $i_K(\lambda_V) = \lambda_K$ and $i_K(\mu_V) = \mu_K$. Then $P(K) := i_K \circ P\colon S^1 \to S^3$.

Given a pattern $P : S^1 \to V$ of winding number n, we obtain a winding number $(-n)$ pattern P^r by reversing the orientation of S^1 while fixing the orientations of V, λ_V, and μ_V. Observe that $P^r(K) = (P(K))^r$, whereas

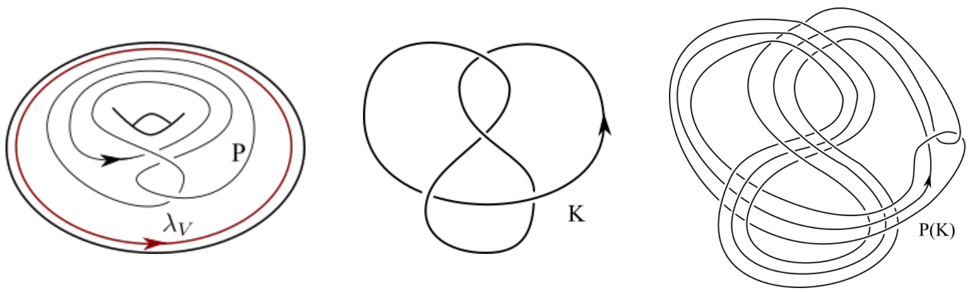

Figure 6.1: A winding number +1 pattern P in the solid torus V with longitude λ_V in red (left), a knot K (center), and the satellite knot $P(K)$ (right).

$P(K^r)$ generally equals neither $P(K)$ nor $P^r(K)$. Our need for this plethora of orientations on P, λ_V, and K in order to obtain a well-defined knot $P(K)$ is evident even in the simplest case: connected sum is not a well-defined operation on unoriented knots.

The work of Litherland [34] completely describes the Casson-Gordon invariants of a satellite knot; we will only need the following special cases.

Theorem 6.2.4 (Litherland [34])**.** *Let P be a satellite operator, described via a curve γ in the complement of $P(U)$ in S^3. Let $n \in \mathbb{N}$ be a prime power, and suppose that γ has n distinct lifts $\gamma_1, \dots, \gamma_n$ to $\Sigma_n(P(U))$. Then for any knot K there is a canonical, covering transformation invariant isomorphism $\phi\colon H_1(\Sigma_n(P(U))) \to H_1(\Sigma_n(P(K)))$ such that for any prime power order character $\chi\colon H_1(\Sigma_n(P(U)) \to \mathbb{Z}_d$ we have*

$$\sigma_1\, \tau(P(K), \chi \circ \phi^{-1}) = \sigma_1\, \tau(P(U), \chi) + \sum_{i=1}^{n} \sigma_K \left(\omega_d^{\chi(\gamma_i)}\right).$$

Theorem 6.2.5 (Litherland [34])**.** *Let P be a winding number m satellite operator with $P(U) = U$ and suppose $n \in \mathbb{N}$ is a prime power such that $(m, n) = 1$. Then for any knot K there is a canonical, covering transformation* *invariant isomorphism* $\phi\colon H_1(\Sigma_n(K)) \to H_1(\Sigma_n(P(K)))$ *such that for any prime power order character* $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ *we have*

$$\sigma_1\,\tau(P(K), \chi\circ\phi^{-1}) = \sigma_1\,\tau(K,\chi).$$

6.3 Winding number m and $-m$ patterns are unbounded distance in their action on concordance.

Let $C_{m,1}$ denote the $(m,1)$ cabling pattern and $C^r_{m,1}$ denote the winding number $-m$ pattern obtained by reversing $C_{m,1}$.

Proposition 6.3.1. *Suppose K is a knot such that n-fold branched cover Casson-Gordon signature obstructions show that $g_4(K\#-K^r) > g$. Then for any m which is relatively prime to n we have that $g_4(C_{m,1}(K)\#-C^r_{m,1}(K)) > g$ too.*

Proof. First, observe that by Theorem 6.2.5 we have a canonical, covering transformation invariant correspondence between the Casson-Gordon signatures of K corresponding to the n-fold branched cover and those of $C_{m,1}(K)$. So the n-fold branched cover Casson-Gordon signature obstructions show that $g_4(C_{m,1}(K)\#-(C_{m,1}(K))^r) > g$. But $-(C_{m,1}(K))^r = -(C^r_{m,1}(K))$. □

We will show in Theorem 6.3.3 that for any odd prime p and any $g \in \mathbb{N}$, there exists a knot K such that $g_4(K\#-K^r) > g$ as detected by p-fold cyclic branched cover Casson-Gordon signatures. Once we have this result, Theorem 6.1.3 follows.

Proof of Theorem 6.1.3. Fix $m > 0$. Let P and Q be arbitrary patterns of winding number m and $-m$, respectively. Observe that Theorem 6.1.1 implies that $d(C_{m,1}, P)$ and $d(C^r_{m,1}, Q)$ are both finite, so it suffices to show that $d(C_{m,1}, C^r_{m,1})$ is infinite. Let $g \geq 0$ be given, and let p be an odd prime which does not divide m. By Theorem 6.3.3, there exists a knot K such that the pth cyclic branched cover Casson-Gordon signatures show that $g_4(K \# -K^r) > g$. By Proposition 6.3.1, we therefore have that $g_4(C_{m,1}(K) \# -C^r_{m,1}(K)) > g$, too. $\square$

For a fixed p and g, we will take $K = \#^{g+1} J_g$, where J_g is obtained by iterated satellite operations along a $(p-1)$-component unlink $\{\eta_j\}_{j=1}^{p-1}$ in the complement of some knot J_0. The key property of J_0 will be that for some prime q and distinct $a_1, \dots, a_{p-1} \in \mathbb{F}_q$,

$$H_1(\Sigma_p(J_0), \mathbb{F}_q) \cong \mathbb{F}_q[t]/\langle \Phi_p(t) \rangle \cong \bigoplus_{j=1}^{p-1} \mathbb{F}_q[t]/\langle t - a_j \rangle.$$

(Here $\Phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$.) Each curve η_j will correspond to a generator of the $\mathbb{F}_q[t]/\langle t - a_j \rangle$- summand above, in a way we will make precise.

Proposition 6.3.2. *For any odd prime p, there exists a prime q and a knot J such that*

$$H_1(\Sigma_p(J), \mathbb{F}_q) \cong \mathbb{F}_q[t]/\langle \Phi_p(t) \rangle \cong \bigoplus_{j=1}^{p-1} \mathbb{F}_q[t]/\langle t - a_j \rangle,$$

where $a_1, \dots, a_{p-1}$ are distinct elements of $\mathbb{F}_q$ and $a_{p-j} \equiv a_j^{-1} \mod q$.

Proof. Let q be a prime which is equivalent to 1 mod p, so we can write $q = kp + 1$ for some $k \in \mathbb{N}$. Note that $0 < k < q$ and so k is a unit mod q. Let $a(t) = k\Phi_p(t) - qt^{\frac{p-1}{2}}$. Observe that $a(t)$ is a symmetric polynomial with $a(1) = kp - q = -1$. Levine's work [29] characterizing the Alexander polynomials of knots implies that there is a knot with Alexander polynomial equal to $a(t)$. In fact, his construction gives a knot J with Alexander module given by $H_1(\widetilde{X_J}) \cong \mathbb{Z}[t,t^{-1}]/\langle a(t)\rangle$, so

$$\begin{aligned}H_1(\Sigma_p(J),\mathbb{F}_q) \cong H_1(\widetilde{X_J},\mathbb{F}_q)/\langle t^p-1\rangle \cong \mathbb{F}_q[t]/\langle a(t),t^p-1\rangle \\ \cong \mathbb{F}_q[t]/\langle k\Phi_p(t),t^p-1\rangle \cong \mathbb{F}_q[t]/\langle \Phi_p(t)\rangle.\end{aligned}$$

It is a standard fact of number theory that since the order of q mod p is 1, $\Phi_p(t)$ splits into linear factors over $\mathbb{F}_q$. In addition, $\Phi_p(t)$ has distinct roots, as one can easily verify by considering $f(t) = (t-1)\Phi_p(t) = t^p - 1$. Since the only root of $f'(t) = pt^{p-1}$ over $\mathbb{F}_q$ is $t = 0$, we have that $f'(t)$ and $f(t)$ have no common roots and so $f(t)$ has no repeated roots over $\mathbb{F}_q$. So $\Phi_p(t)$ certainly cannot have repeated roots either and there are distinct $a_1,\ldots,a_{p-1} \in \mathbb{F}_q$ such that

$$H_1(\Sigma_p(J),\mathbb{F}_q) \cong \mathbb{F}_q[t]/\langle \Phi_p(t)\rangle \cong \bigoplus_{j=1}^{p-1} \mathbb{F}_q[t]/\langle t-a_j\rangle.$$

Note that this decomposition is canonical, since the $\mathbb{F}_q[t]/\langle t-a_j\rangle$ summand is exactly the eigenspace of the action of t corresponding to eigenvalue a_j. Since $\Phi_p(a) = 0$ if and only if $\Phi_p(a^{-1}) = 0$, after reordering we can also assume that $a_{p-j} \equiv a_j^{-1} \mod q$. □

Now fix an odd prime p and let J_0 be as in Proposition 6.3.2. For $j = 1,\ldots,p-1$, let $x_j \in H_1(\Sigma_p(J_0),\mathbb{F}_q)$ be an arbitrary generator of the $\mathbb{F}_q[t]/\langle t-a_j\rangle$ summand (i.e., x_j is an eigenvector of the covering transformation induced action on $H_1(\Sigma_p(J_0),\mathbb{F}_q)$ with eigenvalue a_j). Choose elements $\alpha_j \in \pi_1(X_p(J_0)) \subseteq \pi_1(X(J_0))$ which map to x_j under the natural map $\pi_1(X_p(J_0)) \to \pi_1(\Sigma_p(J_0)) \to H_1(\Sigma_p(J_0)) \to H_1(\Sigma_p(J_0)),\mathbb{F}_q)$. Now choose curves $\eta_1,\ldots,\eta_{p-1}$ in the complement of J_0 such that η_j represents a_j in $\pi_1(X(J_0))$ for each $j = 1,\ldots,p-1$. Notice that changing the crossings of the η_j curves with each other does not change this property, so by crossing changes we can assume that $\cup_{j=1}^{p-1}\eta_i$ is an unlink in S^3. For a choice of knots $A_1, \ldots, A_{p-1}$, denote by $J(A_1, \ldots, A_{p-1})$ the knot obtained by infecting J_0 by A_i along η_j for $j = 1, \ldots p-1$. Note that since $\cup_{j=1}^{p-1}\eta_i$ is an unlink we can consider this infection as a $(p-1)$-fold iterated satellite operation and Theorem 6.2.4 applies. In particular, observe that for each j the homology classes of the p lifts of η_j are given by $\{t^k x_j = a_j^k x_j\}_{k=1}^{p}$. Theorem 6.2.4 then implies that given any character $\chi\colon H_1(\Sigma_p(J_0)) \to \mathbb{Z}_q$, under the natural identification of $H_1(\Sigma_p(J(A_1, \ldots A_{p-1})))$ with $H_1(\Sigma_p(J_0))$ we have

$$\sigma_1\, \tau(J(A_1, \ldots A_{p-1}), \chi) = \sigma_1\, \tau(J_0, \chi) + \sum_{j=1}^{p-1} \left[\sum_{k=1}^{p} \sigma_{A_j}\left(\omega_q^{\chi(t^k x_j)}\right) \right] \tag{6.1}$$

By the proof of Theorem 1 of Cha-Livingston [5], for any $\omega \in S^1$ there is some $\omega' \in S^1$ arbitrarily close to ω and a knot K whose jumps in the Tristram-Levine signature function occur exactly at ω' and $\overline{\omega'}$. In particular, there exists a knot C such that the only jumps in the Tristram-Levine signature $\sigma_C(\omega)$ occur just before ω_q and just after $\overline{\omega_q}$ and hence one such that

$$\sigma_C(\omega_q^k) = \begin{cases} 0 & k \equiv 0 \mod q \\ \sigma_C(\omega_q) > 0 & k \not\equiv 0 \mod q \end{cases}.$$

Now also fix $g \geq 0$. Let $c = \max\limits_{\chi\colon H_1(\Sigma_p(J_0)) \to \mathbb{Z}_q} \{|\,\sigma_1\, \tau(J_0, \chi)|\}$. By taking sufficiently large connected sums of C, we can obtain knots A and B such that for all $1 \leq i \leq q-1$ we have

$$\begin{aligned} p\,\sigma_A(\omega_q^i) &= p\,\sigma_A(\omega_q) > 2(g+1)c + 2pg \\ p\,\sigma_B(\omega_q^i) &= p\,\sigma_B(\omega_q) > (g+1)p\,\sigma_A(\omega_q) + 2(g+1)c + 2pg. \end{aligned}$$

Recall that for any $1 \leq j \leq p-1$ and $1 \leq k \leq p$, we have $t^k x_j = a_j^k x_j$ for some nonzero eigenvalue a_j. It follow that $\chi(t^k x_j) = \chi(a_j^k x_j) = a_j^k \chi(x_j)$ is congruent to 0 mod q if and only if $\chi(x_j) \equiv 0 \mod q$. It follows that for $1 \leq j \leq p-1$ we have $\sum_{k=1}^{p} \sigma_A\left(\omega_q^{\chi(t^k x_j)}\right) = \begin{cases} p\,\sigma_A(\omega_q) & \text{if } \chi(x_j) \not\equiv 0 \mod q \\ 0 & \text{if } \chi(x_j) \equiv 0 \mod q \end{cases}$, as well as an analogous formula for B.

We choose $A_1 = A_2 = \cdots = A_{(p-1)/2} = A$ and $A_{(p+1)/2} = \cdots = A_{p-1} = B$ and let $J_{A,B} = J(A_1, \ldots, A_{p-1})$. The key point here is that since $a_j^{-1} \equiv a_{p-j}$ mod q for all j, we infect curves corresponding to eigenvalues a and a^{-1} with different knots.

Now define $\delta_j(\chi) = \begin{cases} 1 & \text{if } \chi(x_j) \not\equiv 0 \mod q \\ 0 & \text{if } \chi(x_j) \equiv 0 \mod q \end{cases}$, and observe that Equation 6.1 becomes

$$\sigma_1\,\tau(J_{A,B}, \chi) = \sigma_1\,\tau(J_0, \chi) + p\,\sigma_A(\omega_q) \sum_{j=1}^{\frac{p-1}{2}} \delta_j(\chi) + p\,\sigma_B(\omega_q) \sum_{j=\frac{p+1}{2}}^{p-1} \delta_j(\chi) \quad (6.2)$$

Theorem 6.3.3. *For fixed odd p and $g \geq 0$, let $J_g = J_{A,B}$ be as above and let $K = \#^{g+1} J_g$. Then the Casson-Gordon signatures associated to the pth cyclic branched cover show that $g_4(K \# -K^r) > g$.*

Proof. As in Proposition 6.2.1, our identification

$$H_1(\Sigma_p(J_g), \mathbb{F}_q) \cong H_1(\Sigma_p(J_0), \mathbb{F}_q) \cong \bigoplus_{j=1}^{p-1} \left(\mathbb{F}_q[t]/\langle t - a_j \rangle\right) \langle x_j \rangle$$

induces a description

$$H_1(\Sigma_p(-J_g^r), \mathbb{F}_q) \cong \bigoplus_{j=1}^{p-1} \left(\mathbb{F}_q[t]/\langle t^{-1} - a_j \rangle\right) \langle y_j \rangle,$$

such that $\sigma_1\,\tau(-J_g^r, \chi\colon y_j \mapsto c_j) = -\,\sigma_1\,\tau(J_g, \chi\colon x_j \mapsto c_j)$. We therefore have that

$$H_1(\Sigma_p(K\# -K^r), \mathbb{F}_q) \cong \bigoplus_{i=1}^{g+1} H_1(\Sigma_p(J_g), \mathbb{F}_q) \oplus \bigoplus_{i=1}^{g+1} H_1(\Sigma_p(-J_g^r), \mathbb{F}_q) \tag{6.3}$$

$$= \bigoplus_{i=1}^{g+1} \bigoplus_{j=1}^{p-1} (\mathbb{F}_q[t]/\langle t-a_j\rangle)\,\langle x_j^i\rangle \oplus \bigoplus_{i=1}^{g+1} \bigoplus_{j=1}^{p-1} \left(\mathbb{F}_q[t]/\langle t^{-1}-a_j\rangle\right)\langle y_j^i\rangle \tag{6.4}$$

For any $\chi = \bigoplus_{i=1}^{g+1} \chi_i \oplus \bigoplus_{i=1}^{g+1} \chi_i'$ and for $1 \le j \le p-1$, define $n_j(\chi)$ and $n_j'(\chi)$ as follows: $n_j(\chi) = \sum_{i=1}^{g+1} \delta_j(\chi_i)$ and $n_j'(\chi) = \sum_{i=1}^{g+1} \delta_j(\chi_i')$. By the additivity of Casson-Gordon signatures and Equation 6.2, we have the following formula for $(*) = \sigma_1\, \tau(K\# -K^r, \chi)$:

$$\begin{aligned}
(*) &= \sum_{i=1}^{g+1} \sigma_1\, \tau(J, \chi_i) + \sum_{i=1}^{g+1} \sigma_1\, \tau(-J^r, \chi_i') \\
&= \sum_{i=1}^{g+1} \left(\sigma_1\, \tau(J_0, \chi_i) + \sum_{j=1}^{p-1} \delta_j(\chi_i) p\, \sigma_{A_j}(\omega_q) \right) \\
&\qquad\qquad - \sum_{i=1}^{g+1} \left(\sigma_1\, \tau(J_0, \chi_i') + \sum_{j=1}^{p-1} \delta_j(\chi_i') p\, \sigma_{A_j}(\omega_q) \right).
\end{aligned}$$

Our definition of $n_j(\chi)$ and $n_j'(\chi)$ then gives that

$$\begin{aligned}
(*) &= \sum_{i=1}^{g+1} \left(\sigma_1\, \tau(J_0, \chi_i) - \sigma_1\, \tau(J_0, \chi_i')\right) + \sum_{j=1}^{p-1} \left(n_j(\chi) - n_j'(\chi)\right) p\, \sigma_{A_j}(\omega_q) \\
&= \sum_{i=1}^{g+1} \left(\sigma_1\, \tau(J_0, \chi_i) - \sigma_1\, \tau(J_0, \chi_i')\right) + p\, \sigma_A(\omega_q) \sum_{j=1}^{\frac{p-1}{2}} \left(n_j(\chi) - n_j'(\chi)\right) \\
&\qquad\qquad + p\, \sigma_B(\omega_q) \sum_{j=\frac{p+1}{2}}^{p-1} \left(n_j(\chi) - n_j'(\chi)\right).
\end{aligned}$$

Note that $H_1(\Sigma_p(K\# -K^r), \mathbb{F}_q)$ is isomorphic as a group to $\mathbb{F}_q^{(p-1)(2g+2)}$, so a subgroup H has rank r if and only if it has order q^r. We wish to apply

Theorem 6.2.2 to conclude that $g_4(K\# - K^r) > g$. It is easy to check that it suffices to prove the following claim.

Claim: For every covering transformation invariant subgroup $H \leq H_1(\Sigma_p(K\#-K^r), \mathbb{F}_q)$ of rank $(p-1)(2g+1)$ there exists $\chi\colon H_1(\Sigma_p(K\# - K^r)) \to \mathbb{F}_q$ which vanishes on H such that $|\,\sigma_1\,\tau(K\# - K^r, \chi)| > 2pg$.

Let H be as in the claim. Since H is an invariant subspace and $H_1(\Sigma_p(K\#-K^r), \mathbb{F}_q)$ is spanned by eigenvectors, H has a basis of eigenvectors β', as proven for instance in [26]. Let B_j be the a_j-eigenspace of the covering transformation induced action on $H_1(\Sigma_p(K\# - K^r), \mathbb{F}_q)$, for $1 \leq j \leq p-1$. Note B_j has a basis $\beta_j = \{x_j^i\}_{i=1}^{g+1} \sqcup \{y_{p-j}^i\}_{i=1}^{g+1}$ and is rank $2g+2$. Since H is spanned by eigenvectors, we have that

$$\sum_{j=1}^{p-1} \operatorname{rank}(B_j \cap H) = \operatorname{rank}(H) = (p-1)(2g+1).$$

Since $H \neq H_1(\Sigma_p(K\# - K^r), \mathbb{F}_q)$ there is some j_0 such that $B_{j_0} \not\subset H$. Assume without loss of generality that $j_0 \leq \frac{p-1}{2}$. Let v_1 be in B_{j_0} but not in H. We can extend $\beta' \sqcup \{v_1\}$ to a basis β'' of eigenvectors for $H_1(\Sigma_p(K\#-K^r), \mathbb{F}_q)$ by adding some $p-2$ vectors, $v_2, \ldots, v_{p-1}$. Let χ be defined as follows on elements of β'' and then extended linearly over $H_1(\Sigma_p(K\# - K^r), \mathbb{F}_q)$:

$$\chi(v) = \begin{cases} 0 & \text{if } v \in \beta' \\ 1 & \text{if } v = v_1 \\ 0 & \text{if } v = v_i \text{ for } i = 2, \ldots, p-1 \end{cases}.$$

Observe that χ vanishes both on H and on B_j for all $j \neq j_0$. We therefore have that $n_j(\chi) = 0$ for $j \neq j_0$ and $n'_j(\chi) = 0$ for $j \neq p - j_0$. Our formula for

$\sigma_1\,\tau(K\# - K^r, \chi)$ therefore becomes

$$\sigma_1\,\tau(K\# - K^r, \chi) = \sum_{i=1}^{g+1} (\sigma_1\,\tau(J_0, \chi_i) - \sigma_1\,\tau(J_0, \chi_i')) + p\,\sigma_A(\omega_q) n_{j_0}(\chi)$$
$$- p\,\sigma_B(\omega_q) n'_{p-j_0}(\chi).$$

Since χ is not the zero character we must have that one of $n_{j_0}(\chi)$ and $n'_{p-j_0}(\chi)$ is positive.

Case 1: $n'_{p-j_0}(\chi) > 0$. Then, noting that $n_{j_0}(\chi) \leq g+1$, by our choice of $\sigma_B(\omega_q)$ we have

$$\sigma_1\,\tau(K\# - K^r, \chi) \leq 2(g+1)c + p\,\sigma_A(\omega_q)(g+1) - p\,\sigma_B(\omega_q) < -2pg.$$

Case 2: $n'_{p-j_0}(\chi) = 0$ and $n_{j_0}(\chi) > 0$. Then by our choice of $\sigma_A(\omega_q)$ we have

$$\sigma_1\,\tau(K\# - K^r, \chi) \geq -2(g+1)c + p\,\sigma_A(\omega_q) > 2pg. \qquad \square$$

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